NBS

Technical Note

No. 211
Volume 1

CONFERENCE ON NON-LINEAR PROCESSES IN THE IONOSPHERE DECEMBER 16-17, 1963

EDITORS

DONALD H. MENZEL AND ERNEST K. SMITH, JR.

Sponsored By

Voice of America

and

Central Radio Propagation Laboratory
National Bureau of Standards
Boulder, Colorado

U. S. DEPARTMENT OF COMMERCE
NATIONAL BUREAU OF STANDARDS

NATIONAL BUREAU OF STANDARDS

Technical Note 211, Volume 1

Issued April 17, 1964

CONFERENCE ON NON-LINEAR PROCESSES IN THE IONOSPHERE

DECEMBER 16-17, 1963

Sponsored by
Voice of America
and
Central Radio Propagation Laboratory
National Bureau of Standards
Boulder, Colorado

NBS Technical Notes are designed to supplement the Bureau's regular publications program. They provide a means for making available scientific data that are of transient or limited interest. Technical Notes may be listed or referred to in the open literature.

For sale by the Superintendent of Documents, U. S. Government Printing Office
Washington, D.C. 20402
Price 50¢

PREFACE

This Conference was conceived by Professor Donald H. Menzel about a year ago. In view of Dr. Ernest K. Smith's concern with the effects of high-power transmissions in the ionosphere stemming from his association with the Voice of America, I asked the two of them to act as Co-chairmen for this Conference.

These gentlemen, together with an Advisory Committee made up of Professor David Layzer of Harvard, George Jacobs of the Voice of America and Dr. James R. Wait, Roger Gallet and myself from the Central Radio Propagation Laboratory have been responsible for the technical planning. Robert T. Frost was in charge of local arrangements and also served as Secretary/Treasurer for the Conference.

The non-technical editorial work was carried out under the supervision of Mrs. Mildred F. Talbutt, ably assisted by Mrs. Dorothy M. Burdick and Mrs. Anna M. von Kreisler. Dr. J. Robert Lebsack of the Technical Information Office has contributed very usefully to the planning of the publication phase.

Most of the papers presented at the meeting are published in the six volumes of this Technical Note No. 211. Primary responsibility for their technical content must rest, of course, with the individual authors and their organizations.

C Gordon Little

C. Gordon Little
Director, Central Radio
Propagation Laboratory

Volume I

Table of Contents

Additional papers for Session 1 continued in Volume II.

Other Papers Presented at the Conference on Non-Linear Processes in the Ionosphere*

---

*Copies of these particular papers are not available from the Conference Organization; however, in some cases they may be obtained directly from the authors at the addresses indicated.

Complete Contents

THE GENERAL PROBLEM OF IONOSPHERIC NONLINEARITIES

Donald H. Menzel[1]

I wish, first of all, to extend a welcome on behalf of Dr. Ernest K. Smith and myself, co-chairmen of this conference. My task this morning would have been much lighter if only two distinguished Soviet scientists, Prof. V. L. Ginsburg and Prof. A. V. Gurevich, had been able to accept our invitation. Their joint paper [1] on "Nonlinear Phenomena in a Plasma Located in an Alternating Electromagnetic Field" is a classic. It reviews and extends the basic principles and applies them to an examination of general non-linear problems.

Their paper is far too long and detailed for me to review here. Perhaps I should just commend it to your attention and then sit down. But such brevity is hardly in the tradition of conference chairmen. Besides, I should like to discuss some relatively unfamiliar non-linear processes that may occur in certain types of ionized plasmas.

Various contributors to this conference have asked me to explain the qualifying adjective, "non-linear," which defines the type of processes we are discussing here today. Let me say, first of all, that "non-linear" does not signify "irregular" or"non-uniform." The mathematician, the physicist, and the engineer each has his own definition of "non-linear." At first sight these explanations may seem to be very different,

---

[1] Harvard College Observatory, Cambridge, Mass.

but they are actually only alternate ways of expressing the same basic facts.

To the mathematician, quantities such as E, $\frac{dE}{dt}$ $\frac{d^2E}{dt^2}$, etc., are linear in that they depend only on the first power of E. Differential equations involving these quantities have solutions of the form:

$$E = f(t) .$$

And if two solutions are simultaneously present, the complete solution is simply the sum of the two, or

$$E = f_1(t) + f_2(t) .$$

Mathematicians call this the principle of superposition.

Mathematical quantities involving the square, or higher power, or cross-product of the foregoing, such as E^2, $\left(\frac{dE}{dt}\right)^2$, $E \frac{dE}{dt}$, etc., are non-linear. Mathematically, we recognize the non-linearity because the complete solution is not the sum of two elementary solutions. Cross-product terms exist. The two disturbances react on one another. For example,

$$E^2 = f_1^2(t) + 2f_1(t) f_2(t) + f_2^2(t) ,$$

and so on.

Physically, E may represent an electric field that varies with the time. Any number of electromagnetic fields may exist simultaneously in any medium. As long as the disturbances are linear, we may represent their total effect as

the sum of the individual disturbances. But if the phenomenon is non-linear, the total effect may depend on the cross-product of the individual disturbances.

The engineer has a practical definition of non-linear phenomena. As long as a simple filter can separate the different superposed periodic disturbances passing through a circuit, they are linear. But when the two or more disturbances have interacted in such a way that they cannot be so separated, then the phenomenon is non-linear. Under certain circumstances where he wishes to induce artificial mixing, the engineer will use a non-linear device such as a rectifier or detector.

Many of the differential equations of mathematical physics are linear in the first-order. They may contain, however, certain non-linear terms that are negligible unless the variable is exceptionally large. Consider the following equation:

$$\frac{d^2E}{dt^2} + \omega^2 E + \beta E^2 = 0 ,$$

where β is small. When β is zero, this equation has the solution:

$$E = A \sin \omega t.$$

But when we substitute this into the original equation, the term in E becomes

$$\beta E^2 = \beta A^2 \sin^2 \omega t = \frac{\beta A^2}{2} (1-\cos 2\omega t).$$

In other words, the perturbations has caused the harmonic of frequency 2ω to appear.

To obtain a general solution, we may employ the complex Fourier series:

$$E = \sum_{n=-\infty}^{\infty} A_n e^{i\omega nt} .$$

Substituting this into the differential equation, we get

$$\sum A_n \omega^2 (n^2-1) e^{i\omega nt} + \beta \sum \sum A_m A_{m'} e^{i\omega (m+m')t} = 0 .$$

Carry out the double summation by first summing over pairs of m and m' such that

$$m + m' = n .$$

Then we can write

$$\sum_n [A_n \omega^2 (n^2-1) + \beta \sum_m A_m A_{n-m}] e^{i\omega nt} = 0 .$$

Hence we must have

$$A_n \omega^2 (n^2-1) + \beta \sum_m A_m A_{n-m} = 0 .$$

This infinite set of simultaneous equations is, of course, nonlinear in the A_n's. The solution contains higher harmonics, though they may have progressively smaller amplitudes.

The most familiar non-linear ionospheric process has received the name Luxembourg Effect, because it first showed up in connection with radio Luxembourg. The Luxembourg Effect is a particular example of a general phenomenon that might

be more accurately termed "The Interaction of Acoustic and Electromagnetic Waves in an Ionized Medium." The medium may or may not contain a magnetic field.

In the Luxembourg Effect, a radio transmitter of high power produces an intense electromagnetic field in the ionosphere. If an audio frequency modulates the carrier, the electron velocity field responds to this modulation. In effect, the temperature of the electron gas fluctuates by an appreciable amount, according to the impressed audio frequency. This temperature variation may also be described as a pressure variation, a true acoustic wave. I shall refer to the initial modulated wave and the acoustic disturbance it produces in the medium as the "unwanted wave."

The various propagation constants that depend on temperature or pressure of the electron gas in the medium vary with the frequency of the acoustic disturbance. If the density of the gas is high enough to make the medium dissipative, the imaginary part of the complex index of refraction also varies. A second radio wave, termed the "wanted wave," traversing the medium, will suffer from this variable absorption. Hence the wanted wave acquires some of the modulation of the unwanted wave. Since this superposed modulation constitutes an interference with the wanted wave, it is undesired or "unwanted." Thus the terminology.

The theory of this wave interaction was first given by Martyn and Bailey[2], who pointed out the importance of magnetic fields, especially for radio waves near the gyrc frequency.

The formulas they derived, however, were over-simplified, in that they linearized the equations describing the interaction. As a result, their formulas do not properly display the phase lag of the acoustic disturbance relative to the modulation by an amount dependent on the audio frequency. Menzel and Layzer have developed a more detailed theory which will be presented at this conference.

Experiments have shown that the Luxembourg variety of wave interaction rarely leads to cross modulation in excess of a few per cent. The amount is very sensitive to a number of parameters. In the medium itself the significant variables are: the electron density and temperature, the frequency of dissipative electron collisions, and the magnetic field that fixes the electron gyrofrequency. The phenomenon further depends on the power and polarization, in addition to the radio and audio frequencies of the unwanted wave. The frequency and polarization of the wanted wave are also significant. The angle of incidence on the ionosphere determines the degree of penetration. Since the unwanted acoustic disturbance is localized, the greatest interaction will occur when the wanted waves travel through a significant portion of the disturbed volume.

Although the original theory and observations refer to the interaction of two separate and independent waves, examination of the basic equation shows that a single modulated wave in the medium may interact with itself. As before, the

fluctuating field causes an acoustic disturbance in the medium. This results in a variable attenuation coefficient, which reacts on the original wave. Since the phase lag is a function of the acoustic frequency, the degree of attenuation will similarly vary with the acoustic frequency. In effect, some frequencies will experience demodulation whereas others will be excessively modulated.

The foregoing phenomenon, termed "self interaction," indicates that any acoustic disturbance - of whatever origin - can impress its modulation on a wave traversing the volume. Rockets, missiles, explosions, collisions of solar ion clouds with the upper atmosphere, etc., represent potential sources of such acoustic waves. The question requiring study is the magnitude of the interaction, not whether such interaction occurs. More specifically, of the various kinds of interaction, are there any that could be studied experimentally?

Certain difficulties are obviously present, resulting from fundamental differences in acoustic fields of radio origin on one hand and of mechanical origin on the other. For example, since the velocity of propagation of a radio wave greatly exceeds that of sound, the acoustic field induced by a modulated electromagnetic wave will be in phase over a large volume of space. Hence the interaction with the wanted wave will likewise be in phase over a large volume of space.

A mechanical disturbance, on the contrary, propagates with a much smaller velocity. Hence a wanted wave, progressing

through the perturbed medium, will suffer high attenuation in certain zones and low attenuation in others. The effect resembles that of interference on optical frequencies. The problem is complex and the total effect on the wanted wave depends on the character and the extent of the basic disturbance.

The details of the interaction require precise analysis. The limited volume of the mechanical disturbance may well be offset by the intensity of the disturbance itself. In a region where shock waves may exist, numerous non-linear effects may occur, complicating the transport of electromagnetic waves through the medium. The complex (dissipative) index of refraction will probably contain significant velocity-dependent terms of non-linear character. The effect on a radio wave transversing the medium may still be appreciable. Magnetohydrodynamic compression of the gas will significantly alter the magnetic field, with attendant complication of the mathematical problem.

M. Cutolo has demonstrated experimentally the existence of a phenomenon sufficiently different from the Luxembourg Effect to deserve a special designation. He refers to it as the "detection effect," since it depends on non-linear ionospheric properties for its occurrence. I prefer to call it the "Cutolo Effect," after its discoverer. I shall describe it briefly, since Cutolo plans to devote his paper to more conventional non-linear problems.

Cutolo directs a beam of pulsed, modulated VHF waves upon the ionosphere. Of significance is his use of the

gyrofrequency as the modulation frequency. In the Luxembourg terminology, this pulsed, modulated, VHF wave, further modulated at audio frequencies to carry intelligence, represents the "unwanted wave." Cutolo reports that a second HF wave, the wanted wave, reflected from the ionospheric region traversed by the intense, focussed VHF signal, will have the audio signal transferred to it. The amount of this cross modulation is very sensitive to the exactness of match between the gyro and modulating frequencies. Cutolo has found that the effect occurs for both vertical and oblique incidence of the wanted wave.

I have not seen, up to this time, any quantitative physical explanation of the Cutolo effect. An unwanted wave on the gyrofrequency would produce a large interaction of the Luxembourg type. But a VHF wave, modulated at the gyrofrequency, is not equivalent to a pure electromagnetic wave on the gyrofrequency. As is well known, such a wave can be expressed as an unmodulated carrier and two unmodulated sidebands separated from the carrier by the gyrofrequency. To account for the Cutolo effect we must suppose that some unknown ionospheric non-linearity so mixes the carrier and sidebands as to release the true gyrofrequency. It is difficult to see, though, how such interactions occur because of the very high frequencies employed. Cutolo used a carrier of the order of 50 megacycles per second.

. . . . . .

The following analysis will serve as an elementary introduction to ionospheric theory for younger scientists. The paper also considers various non-linear effects. An electron of mass m_i and charge $-\varepsilon_i$, moving with vector velocity $\underset{\sim}{v}_i$ in an electromagnetic field $\underset{\sim}{E}$ and $\underset{\sim}{H}$ and external acceleration $\underset{\sim}{F}$, obeys the well-known equation of motion:

$$m_i \frac{d\underset{\sim}{v}_i}{dt} = -\varepsilon_i \underset{\sim}{E} - \frac{\varepsilon_i}{c} \underset{\sim}{v}_i \times \underset{\sim}{H} + m_i \underset{\sim}{F} \ . \tag{1}$$

This equation applies along with Maxwell's equations to define the physical state of the medium and its interaction with the field.

The field $\underset{\sim}{F}$ represents several varieties of mechanical actions. In a macroscopic problem involving the entire atmosphere, $\underset{\sim}{F}$ will include the gravitational acceleration g. It also represents the collisional effects of nearby molecules, which may act as a sort of resistance proportional to the velocity and collision frequency ν. As we shall note later on, this collisional term may be anisotropic, because of pressure gradients in the gas.

Maxwell's equations are:

$$\underset{\sim}{\text{curl}}\ \underset{\sim}{H} = \underset{\sim}{\nabla} \times \underset{\sim}{H} = \frac{\kappa}{c} \frac{\partial \underset{\sim}{E}}{\partial t} + \frac{4\pi}{c} \underset{\sim}{J} \ , \tag{2}$$

$$\underset{\sim}{\text{curl}}\ \underset{\sim}{E} = \underset{\sim}{\nabla} \times \underset{\sim}{E} = -\frac{\mu}{c} \frac{\partial \underset{\sim}{H}}{\partial t} \ , \tag{3}$$

$$\text{div}\ \underset{\sim}{H} = \underset{\sim}{\nabla} \cdot \underset{\sim}{H} = 0 \ , \tag{4}$$

$$\mathrm{div}\ \underset{\sim}{E} = \underset{\sim}{\nabla} \cdot \underset{\sim}{E} = 4\pi\rho_\varepsilon/\kappa \quad , \tag{5}$$

wherein $\underset{\sim}{J}$ is the electric current, ρ_ε the density of electric charge, κ the dielectric constant and μ the magnetic permeability. Simultaneous solution of these equations leads to

$$\nabla^2 \underset{\sim}{E} = \frac{\mu\kappa}{c^2} \frac{\partial^2 \underset{\sim}{E}}{\partial t^2} + \underset{\sim}{\nabla}\, \underset{\sim}{\nabla} \cdot \underset{\sim}{E} \ , \tag{6}$$

$$\nabla^2 \underset{\sim}{H} = \frac{\mu\kappa}{c^2} \frac{\partial^2 \underset{\sim}{H}}{\partial t^2} \tag{7}$$

When ρ_ε is constant, the last term of (6) vanishes and we have the ordinary wave equations. For a plane wave, polarized in the z x plane and traveling along the z axis, we have

$$\underset{\sim}{E} = \underset{\sim}{i}\, E_o\, e^{\pm i\omega(t - nz/c)}, \tag{8}$$

where ω is the circular frequency and c/n the velocity of light in the medium. Thus n is the index of refraction. The quantity n may be complex. The unit vectors $\underset{\sim}{i}$, $\underset{\sim}{j}$, $\underset{\sim}{k}$ are the customary Cartesian set. This equation satisfies (6) when

$$n^2 = \mu\ \kappa\ . \tag{9}$$

For non-magnetic media, $\mu = 1$. Hence $\kappa = 1$ for free space. Part of our problem involves the determination of the effective κ when the space contains electric currents and charges.

Return, now to equation (1) in the form:

$$m_i \frac{d\underset{\sim}{v}_i}{dt} = -\varepsilon_i \underset{\sim}{E} - \frac{\varepsilon_i}{c} \underset{\sim}{v} \times \underset{\sim}{H} - m_i \nu \underset{\sim}{v}_i . \qquad (10)$$

The first term on the right represents electric forces and the second, magnetic forces. The third term, somewhat schematically, represents the resistance that the charged particle encounters as it moves through the medium. The quantity ν is the number of collisions per second experienced by the moving ion. For convenience we shall drop the subscript i. The electrons, because of their great mobility, are the major contributors to the current density $\underset{\sim}{J}$. Thus:

$$\underset{\sim}{J} = -N \varepsilon \underset{\sim}{v} , \qquad (11)$$

where N is the number of electrons per cm^3. Under different assumptions we shall calculate the variations of $\underset{\sim}{v}$ and $\underset{\sim}{J}$ with the time. We shall then substitute the $\underset{\sim}{J}$, so calculated, into Maxwell's equation (2).

Case I $\quad \underset{\sim}{H} = 0, \nu = 0, \underset{\sim}{E} = \text{const}: \quad \underset{\sim}{v} = -\varepsilon \underset{\sim}{E}t/m . \qquad (12)$

Case II $\quad \underset{\sim}{E} = 0, \nu = 0, \underset{\sim}{H} = \text{const} \qquad (13)$

Here take the derivative of (10) with respect to the time. The resulting equation reduces as follows:

$$\frac{d^2\underset{\sim}{v}}{dt} = -\frac{\varepsilon}{mc}\left(\frac{d\underset{\sim}{v}}{dt} \times \underset{\sim}{H}\right) = \left(\frac{\varepsilon}{mc}\right)^2 (\underset{\sim}{v} \times \underset{\sim}{H}) \times \underset{\sim}{H} \qquad (14)$$

Let $\underset{\sim}{v}$ have two components $\underset{\sim}{v}_p$ parallel to and $\underset{\sim}{v}_s$ perpendicular

(German: senkrecht) to the field. The triple vector product reduces to:

$$(\underset{\sim}{v} \underset{\sim}{\times} \underset{\sim}{H}) \underset{\sim}{\times} \underset{\sim}{H} = H^2 \underset{\sim}{v}_s$$

so that:

$$\frac{d^2 \underset{\sim}{v}_s}{dt^2} = \omega_L^2 \underset{\sim}{v}_s \; ; \; \frac{d^2 \underset{\sim}{v}_p}{dt^2} = 0 \; , \tag{15}$$

and

$$\underset{\sim}{v}_s = \underset{\sim}{v}_{os} \, e^{i \omega_L t} ; \; \underset{\sim}{v}_p = \underset{\sim}{v}_{op} ; \; \omega_L = \varepsilon H/mc. \tag{16}$$

The particle executes a spiral path with the Larmor frequency ω_L. The radius, a, and pitch, p, of the spiral are:

$$a = v_{os}/\omega_L \; , \; p = v_{op}/\omega_L \; . \tag{17}$$

Case III, $\underset{\sim}{E} = 0, \; \underset{\sim}{H} = 0, \; \nu = \text{const}; \; \underset{\sim}{v} = \underset{\sim}{v}_o \, e^{-\nu t}$, (18)

corresponding to simple damping.

Case IV,
$$\left.\begin{aligned} &E = E_o \, e^{i \omega t}, \; \underset{\sim}{H} = 0, \; \nu = 0, \\ &\underset{\sim}{v} = (\varepsilon \underset{\sim}{E}_o / i\omega m) \, e^{i \omega t} \; . \end{aligned}\right\} \tag{19}$$

Case V , At this point, let us consider a more general problem for which we assume an electromagnetic wave traveling parallel to the z-axis. Let E_x and E_y be the vector electric fields and H_x and H_y the variable magnetic fields associated with the radiation. Let the magnetic field of the ionosphere be of magnitude H, in the direction of the propagation.

The general equation of motion (10) breaks up into equations for the three components:

$$\left.\begin{aligned}
\frac{dv_x}{dt} &= -\frac{\varepsilon}{m} E_x - \frac{\varepsilon H}{mc} v_y - \nu v_x \\
\frac{dv_y}{dt} &= -\frac{\varepsilon}{m} E_y + \frac{\varepsilon H}{mc} v_x - \nu v_y \\
\frac{dv_z}{dt} &= -\nu v_z ,
\end{aligned}\right\} \qquad (20)$$

Multiply the second equation by $\underline{i}$ and add to the first, employing (16). The result is

$$\frac{d}{dt}(v_x + iv_y) = (i\omega_L - \nu)(v_x + iv_y) - \frac{\varepsilon}{m}(E_x + iE_y). \quad (21)$$

Maxwell's field equations become:

$$\left.\begin{aligned}
&\frac{\partial E_y}{\partial z} = \frac{1}{c}\frac{\partial H_x}{\partial t}, \qquad \frac{\partial E_x}{\partial z} = -\frac{1}{c}\frac{\partial H_y}{\partial t} \\
&-\frac{\partial H_y}{\partial z} = \frac{\kappa}{c}\frac{\partial E_x}{\partial t} - \frac{4\pi N\varepsilon v_x}{c}, \\
&\frac{\partial H_x}{\partial z} = \frac{\kappa}{c}\frac{\partial E_y}{\partial t} - \frac{4\pi N\varepsilon v_y}{c}.
\end{aligned}\right\} \qquad (22)$$

Multiply the second and fourth of these equations by $\underline{i}$ and

combine as before, to give:

$$\frac{\partial}{\partial z}(E_x + iE_y) = \frac{i}{c}\frac{\partial}{\partial t}(H_x + iH_y)$$

$$\frac{\partial}{\partial z}(H_x + iH_y) = -\frac{\kappa i}{c}\frac{\partial}{\partial t}(E_x + iE_y) + \frac{4\pi i N\varepsilon}{c}(v_x + iv_y). \tag{23}$$

Now introduce the complex quantities:

$$\left.\begin{aligned} v_x + iv_y &= V\, e^{\pm i\omega(t-nz/c)} \\ E_x + iE_y &= E\, e^{\pm i\omega(t-nz/c)} \\ H_x + iH_y &= H\, e^{\pm i\omega(t-nz/c)} \end{aligned}\right\} \tag{24}$$

where V, E, and H are complex constants. With the aid of (24), (21) and (23) become:

$$[i(-\omega_L \pm \omega) + \nu]\, V = -\varepsilon E/m \tag{25}$$

$$inE = H \tag{26}$$

$$i\omega nH = -\kappa\omega E \pm 4\pi i N\varepsilon V, \tag{27}$$

These equations contain 4 complex unknowns: E, H, V, and n. Of these, n is the only one uniquely determined. H and V are, as one would expect, merely proportional to E.

We may solve directly for n^2:

$$n^2 = 1 - \frac{4\pi N\varepsilon^2}{m\omega}\,\frac{1}{\omega \mp \omega_L \mp i\nu}\,, \tag{28}$$

the complex index of refraction. Let

$$n = n_r \mp ik\,, \tag{29}$$

where n_r and n_k are real. Then:

$$n_r^2 - k^2 = 1 - \frac{4\pi N\varepsilon^2}{m\omega}\,\frac{\omega \mp \omega_L}{(\omega \mp \omega_L)^2 + \nu^2}\,, \tag{30}$$

$$2n_r k = \frac{4\pi N\varepsilon^2}{m\omega}\,\frac{\nu}{(\omega \mp \omega_L)^2 + \nu^2}\,. \tag{31}$$

The propagation factor becomes:

$$e^{\pm i\omega(t-nz/c)} = e^{\pm i\omega(t-n_r z/c)-\omega kz/c}\,, \tag{32}$$

The last term of the exponent indicates an amplitude that decreases exponentially with the distance. The effect results from the collisions, which transfer a certain amount of energy to the medium.

At this point, note the basic philosophy behind these elementary derivations of the propagation parameters. We have solved the general equation of motion to determine the velocity of the electron under the influence of external fields of force,

electromagnetic, magnetic, and collisional. Our first objective was the calculation of the current density, J. Then, substituting this back into Maxwell's equation (2), we derived, in effect the dielectric constant κ, which by equation (9) is equal to the square of the complex index of refraction. Equation (2) thus takes the successive forms:

$$\begin{aligned} \operatorname{curl} \mathbf{H} &= \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi \mathbf{J}}{c} \\ &= \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} - \frac{4\pi N \varepsilon \mathbf{v}}{c} = \frac{\kappa}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{i\omega\kappa \mathbf{E}}{c} \end{aligned} \tag{33}$$

since $\mathbf{v}$, $\mathbf{E}$, and their derivations all have the same time dependence, $e^{i\omega t}$.

The physical significance of κ becomes more meaningful, perhaps, if we follow the steps indicated by equation (33). Here we may use the complex form of (2),

$$\begin{aligned} \operatorname{curl} \mathbf{H} &= \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi \mathbf{J}}{c} \\ &= \left(\pm\frac{i\omega E}{c} - \frac{4\pi N\varepsilon V}{c}\right) e^{\pm i\omega(t-nz/c)} \\ &= \pm\, i\omega + 4\pi N\varepsilon^2 \\ &= \left[\pm\frac{i\omega}{c} + \frac{4\pi N\varepsilon^2}{mc}\,\frac{1}{i(-\omega_L \pm \omega)+\nu}\right] E\, e^{\pm i\omega(t-nz/c)} \\ &= \pm\frac{i\omega\kappa}{c} E\, e^{\pm i\omega(t-nz/c)}, \end{aligned} \tag{34}$$

where we have used (25) to express V in terms of E. Now, solving for κ we get

$$\kappa = n^2 = 1 - \frac{4\pi N\varepsilon^2}{\omega}\,\frac{1}{\omega \mp \omega_L \mp i\nu} \tag{35}$$

as before.

Now note some of the assumptions implicit in these derivations of κ. In the strictest sense we should have a different velocity, $\underset{\sim}{v}_i$, for each electron. We would then have evaluated the average $\underset{\sim}{v}$ and $\underset{\sim}{J}$ as follows:

$$\underset{\sim}{J} = -\varepsilon N\underset{\sim}{v} = -\varepsilon\sum \underset{\sim}{v}_i = -\varepsilon\int \underset{\sim}{v} f(v)\,dv, \tag{36}$$

where f(v) is the velocity distribution function for the electron gas.

The solution of this problem is far more difficult than appears at first sight. The function, f = f(v), will generally depend on time and position. The Boltzmann transport equation is basic for the determination of f.

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \underset{\sim}{v}\cdot\underset{\sim}{\nabla} f + \underset{\sim}{\alpha}\cdot\underset{\sim}{\nabla}_v f + S = 0. \tag{37}$$

In the above, $\underset{\sim}{\alpha}$ is the acceleration and $\underset{\sim}{\nabla}_v$ the gradient in velocity space. The quantity, S, is the collision integral, which depends on the rate of change in the distribution function resulting from collisions. This term takes account of the creation or destruction of electrons in a given range of

velocity. To calculate S we must know the physical details of both elastic and inelastic collisions.

The first three terms of (37) are vector abbreviations for the purely mathematical concept of the total derivative of a function, f, that depends on the time, on the coordinates, and the velocity components. In rectangular coordinates, we may write

$$\frac{d}{dt} f(x,y,z,u,v,\omega,t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$+ \frac{\partial f}{\partial u}\frac{du}{dt} + \frac{\partial f}{\partial v}\frac{dv}{dt} + \frac{\partial f}{\partial \omega}\frac{d\omega}{dt} + S = 0 , \qquad (38)$$

wherein u, v, and ω are the three velocity components.

Let $\underset{\sim}{v}'$ and $\underset{\sim}{v}$ be the vector velocities of the electron prior to and after the collision. Let $\underset{\sim}{v}'_1$ and $\underset{\sim}{v}_1$ be the initial and final velocities of the colliding particle. Let $f(\underset{\sim}{v}')d\underset{\sim}{v}'$ and $f(\underset{\sim}{v})d\underset{\sim}{v}$ symbolically represent the number of electrons within the ranges $d\underset{\sim}{v}'$ and $d\underset{\sim}{v}$, and let $F(\underset{\sim}{v}'_1)d\underset{\sim}{v}'_1$ and $F(\underset{\sim}{v}_1)d\underset{\sim}{v}_1$ similarly represent the number of colliding particles in ranges $d\underset{\sim}{v}'_1$ and $d\underset{\sim}{v}_1$.

The relative velocity of collision becomes:

$$\underset{\sim}{u} = \underset{\sim}{v} - \underset{\sim}{v}_1 \quad \text{and} \quad u = |\underset{\sim}{v} - \underset{\sim}{v}_1|. \qquad (39)$$

Let $\sigma(u,\theta)$ be the differential scattering cross-section describing the probability that a colliding electron will be

deflected through an angle θ, during collision. Define the solid angle $d\Omega = \sin\theta d\theta d\phi$. Then we may express S as follows:

$$S = \iint d\underset{\sim}{v}_1 d\Omega \sigma(u,\theta)\ u\ [f(\underset{\sim}{v})F(\underset{\sim}{v}_1) - f(\underset{\sim}{v}')F(v_1')] \tag{40}$$

The problem of calculating S and f depends on a precise knowledge of the target cross section, σ, as a function of u and θ. We shall not consider it further here, except to note that our arbitrary representation of the collisional effects by the term $m\nu\underset{\sim}{v}$ in equation (10) was naive, to say the least. It was particularly an oversimplification in that we assumed ν to be independent of $\underset{\sim}{v}$, in order to obtain a linear equation.

We should have been much more realistic if we had assumed ν varied with the velocity. Certainly the collision frequency will depend on the temperature of the electron gas. This temperature, in turn, will vary with the impressed electric field. And if the field is modulated, the modulation frequency will cause ν to vary at that frequency. Thus the absorption coefficient, k, equation (31), should also vary with the time. It is this variable absorption that leads to cross-modulation, the Luxembourg effect, gyro-interaction and other non-linear responses, some of which we shall be discussing during this conference.

But additional, still less obvious assumptions exist

in our over-simplified theoretical discussion. Maxwell's equations apply, of course. We have ignored possible reactions of the ionospheric plasma with the terrestrial magnetic field. I refer particularly to phenomena lying in the realm of magnetohydrodynamics. We can at least infer the character of some of these problems.

Rewrite equation (1) in the general form:

$$m_i \frac{d\underset{\sim}{v}_i}{dt} = \varepsilon_i \underset{\sim}{E} - \frac{\varepsilon_i \underset{\sim}{v}_i x \underset{\sim}{H}}{c} + m_i \underset{\sim}{F} \, , \tag{41}$$

wherein a positive ε_i refers to an ion and a negative ε_i to an electron. We can take averages over a unit volume as follows:

$$\begin{aligned} \rho &= \Sigma m_i \\ \rho \underset{\sim}{v} &= \Sigma m_i \underset{\sim}{v}_i \\ \underset{\sim}{J} &= \Sigma \varepsilon_i \underset{\sim}{v}_i \\ \rho_\varepsilon &= \Sigma \, \varepsilon_i \end{aligned} \tag{42}$$

wherein ρ is the mass per unit volume. These summations imply, when relevant, integration over the velocity-distribution functions previously discussed.

The force-field $\underset{\sim}{F}$ breaks up into two parts; external fields such as gravity and internal fields which result from colliding particles. One may write the latter force on a given particle, i, as follows:

$$\underset{\sim}{F} \text{ int} = \sum_j \underset{\sim}{F}_{ij} \, , \tag{43}$$

where the summation is taken over all neighboring particles j. This term cancels out only when the medium is uniform. If electric currents are present, the term gives rise to forces simulated in our earlier treatment by the collisional term, $m_i \nu \underset{\sim}{v}$. This force can also be expressed, for the entire medium, as a sort of electrical resistance, R, in the form $R\underset{\sim}{J}$. It also produces such major forces as pressure gradients. Tangential forces, in the presence of sheer, lead to the phenomenon of viscosity. In the presence of temperature gradients it can even cause thermoelectric effect. For example we may write

$$\underset{\sim}{E} = R\,(\underset{\sim}{J} - \theta\,\underset{\sim}{\nabla}\,T), \tag{44}$$

where θ is a thermoelectric coefficient. In the strictest sense both R and θ will be tensors rather than scalars.

As we sum equation (41) over the individual particles and employ Maxwell's equations, we obtain the general equation of magnetohydrodynamic motion:

$$\begin{aligned} \rho \frac{d\underset{\sim}{v}}{dt} &= -\underset{\sim}{\nabla} p + \rho \underset{\sim}{F}_{exp} + \frac{1}{4\pi}(\underset{\sim}{\nabla} \times \underset{\sim}{H}) \times \underset{\sim}{H} \\ &+ \frac{1}{4\pi}(\underset{\sim}{\nabla} \times \underset{\sim}{E}) \times \underset{\sim}{E} + \frac{1}{4\pi}\underset{\sim}{E}\,\underset{\sim}{\nabla}_1 \cdot \underset{\sim}{E} \\ &- \frac{1}{4\pi c}\frac{\partial}{\partial t}(\underset{\sim}{E} \times \underset{\sim}{H})\,. \end{aligned} \tag{45}$$

When H and E are constant or zero, the equation reduces to that of ordinary hydrodynamics, less the second-order terms related to viscosity. The terms represent the forces arising

from the electromagnetic fields. $\underset{\sim}{E} \times \underset{\sim}{H}$ is the Poynting vector, which represents energy flowing out of the volume element. Hence the partial time derivative of $\underset{\sim}{E} \times \underset{\sim}{H}$ represents the forces caused by radiation pressure.

Now multiply (31) by ε/m and sum again. The result is

$$\frac{m}{n\varepsilon^2}\frac{d\underset{\sim}{J}}{dt} = \underset{\sim}{E} + \frac{1}{c}(\underset{\sim}{v} \times \underset{\sim}{H}) - R(\underset{\sim}{J} - \theta\,\underset{\sim}{\nabla}\,T) \qquad (46)$$

where m is the mass of an electron and n the number of electrons per unit volume. In taking this sum we have allowed for the fact that electrons, by virtue of their small mass, are the main contributors to the electric currents. In passing we may note that this equation has several obvious limiting forms. It can reduce to Ohms law

$$\underset{\sim}{E} = R\,\underset{\sim}{J}\,, \qquad (47)$$

or to the electromotive force induced in a conductor moving through a magnetic field

$$\underset{\sim}{E} = -(\underset{\sim}{v} \times \underset{\sim}{H})/c\,. \qquad (48)$$

Even equation (46) has its limitations. Acoustic waves or hydrodynamic waves can directly produce changes in $\underset{\sim}{J}$, by simple compression or expansion of the current-bearing regions. Moreover, in a conducting medium, $\underset{\sim}{H}$ itself may change under magnetohydrodynamic forces. Such changes in $\underset{\sim}{H}$ produce fluctuations in $\underset{\sim}{J}$. As we have just noted, variations in $\underset{\sim}{J}$ directly

affect the dielectric constant. We must, therefore, set up the equations governing the variations, not only of $\underset{\sim}{H}$ but also of $\underset{\sim}{E}$.

The following analysis is by no means complete. However, consider the steady-state situation, derived from (46);

$$\underset{\sim}{E} + (\underset{\sim}{v} \times \underset{\sim}{H})/c - R\,\underset{\sim}{J} = 0 \qquad (49)$$

Take the curl of this equation and reduce by means of the Maxwell relationships. We get:

$$\begin{aligned} \frac{d\underset{\sim}{H}}{dt} &= \frac{\partial \underset{\sim}{H}}{\partial t} + \underset{\sim}{v} \cdot \underset{\sim}{\nabla}\, \underset{\sim}{H} \\ &= \underset{\sim}{H} \cdot \underset{\sim}{\nabla}\, \underset{\sim}{v} - \underset{\sim}{H}\, \underset{\sim}{\nabla} \cdot \underset{\sim}{v} + \frac{c^2 R}{4\pi}\left[\nabla^2 H - \frac{\kappa}{c^2}\frac{\partial^2 \underset{\sim}{H}}{\partial t^2}\right] . \end{aligned} \qquad (50)$$

Now take the partial time derivative of (49) and derive a similar equation for $\underset{\sim}{E}$. The result is:

$$\frac{\partial \underset{\sim}{E}}{\partial t} + \frac{1}{c}\frac{\partial}{\partial t}(\underset{\sim}{v} \times \underset{\sim}{H}) = \frac{c^2 R}{4\pi}\left[\nabla^2 \underset{\sim}{E} - \frac{\kappa}{c^2}\frac{\partial^2 \underset{\sim}{E}}{\partial t^2} - \underset{\sim}{\nabla}\underset{\sim}{\nabla} \cdot \underset{\sim}{E}\right]. \qquad (51)$$

These equations are also subject to the equation of continuity,

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \underset{\sim}{v} \cdot \underset{\sim}{\nabla}\, \rho = -\rho\, \underset{\sim}{\nabla} \cdot \underset{\sim}{v} ,$$

or

$$\frac{\partial \rho}{\partial t} + \underset{\sim}{\nabla} \cdot (\rho \underset{\sim}{v}) = 0 . \qquad (52)$$

For regions where the conductivity is high, $R \to 0$ and we recover the ordinary equations of magnetohydrodynamics. When the conductivity is low, divide through by R. The electromagnetic equations then reduce to the simple wave equations for $\underset{\sim}{E}$ and $\underset{\sim}{H}$. In other words, for media of low conductivity, the hydrodynamic and electromagnetic fields are independent. Otherwise the two fields are coupled. This means that, in effect, we can combine the purely dynamic terms into a non-linear wave equation, in which the dielectric constant is a function of the dynamical variations.

I shall not attempt to carry the analysis further. My primary objective has been to show that non-linear effects abound in this general problem. I should point out one additional fact. The electrical resistance R is closely related to the damping factor k. Since k, in turn, depends on both electrical and dynamical factors, the factor R introduces additional non-linearities. Detailed solutions depend on assumptions concerning the basic physical conditions in the medium. I have especially desired to emphasize the importance of dynamical or magnetohydrodynamical effects on radio propagation.

[1] V. L. Ginsberg and A. V. Gurevich, Usp. Fiz. Nauk **70**, 201-246, Feb. 1960 and 393-428, March 1960 and english translation: Soviet Physics, Uspekhi, **3**, 115-146 and 175-284, 1960.

[2] D. F. Martyn and V. A. Bailey, Phil Mag **18**, 369, 1934.
V. A. Bailey, Nature, **139**, 68, 838, 1937.
V. A. Bailey, Phil. Mag. **23**, 774, 1937.
V. A. Bailey, Nuovo Cimento Suppl. **4**, 1430, 1956.

ON SOME NON-LINEAR PHENOMENA IN THE IONOSPHERIC PLASMA

P. Caldirola and O. De Barbieri

Istituto di Scienze Fisiche dell'Università di Milano, Milano, Italy

In the first part of the paper we calculate and discuss the distribution function of the electrons of a slightly ionized plasma under the influence of an external magnetic field and of an e.m. wave of the type $\underline{E} = \underline{E}_0 \cos \omega t$.

It is shown that, taking into account both elastic and inelastic collisions between electrons and molecules, it is possible to calculate explicitly the mean electronic energy due only to the absorption of the extraordinary wave of the electric field and that it is maximum at the gyromagnetic resonance ($\omega = \omega_H = \frac{e\mathcal{H}}{mc}$). The study of the effects of inelastic collisions shows that in order to have an equal electronic temperature, the intensity of the electric field must be, in the case of inelastic collisions, about 5 times greater than that for elastic collisions only.

In the second part of the paper we calculate, for the wave $\underline{E} = \underline{E}_0 \cos\omega t$, the components of the complex dielectric permittivity tensor, that are given by expressions dependent in a rather complicated way on $\underline{E}_0$. It is shown that if $\underline{E}_0$ is sufficiently small they can be simplified and, using a suitable perturbation technique, it is possible to calculate a non linear dispersion relation. This relation contains explicitly $\underline{E}_0$, and, for $\underline{E}_0 \rightarrow \underline{0}$, becomes the Appleton-Hartree formula. Furthermore, it shows that in the non-linear case, too, the electric field is split up into two components whose propagation is never independent.

In the third part of the paper we calculate and study the electronic distribution function and the complex dielectric permittivity tensor for a wave of the type $\underline{E}=\underline{E}_0[1+\eta\cos(\alpha t+\beta)]\cos\omega t$ with $\omega \simeq \omega_H \gg \alpha$.

Introduction.

The study of non-linear propagation of electromagnetic waves in plasma has recently aroused the interest of many scientists. In general, the methods used in describing the process of propagation of electromagnetic waves in plasma are based on a procedure of linearization of the equations that give the mathematical representation of the physical problem. A very important physical characteristics of the aforesaid non-linear effects rests on the fact that they can be produced by relatively small electric fields. This can be understood quite easily if we consider a slightly ionized plasma in which the electron-electron and electron-ion collision frequency is much less than the electron-molecule collision frequency. In fact, the propagation of an electromagnetic wave in a plasma causes a relatively high increase of the kinetic energy of the electrons. This is mainly due to two facts: the first is the quite large value of the mean free path λ of the electrons in the plasma (so that they can acquire considerable energy from the wave between two collisions), the second is that due to the smallness of the ratio $\delta = \frac{2m}{M} \simeq 3.4 \cdot 10^{-5}$ between the mass of the electron and the mean mass of the molecules, the mean energy transferred in a collision from the electrons to the neutral component of the plasma is almost negligible. As a final result we see that only the energy distribution of the electrons is altered (with an increase of their temperature and mean energy). This variation depends on the quantities $\underline{E}$, $\underline{H}$, ω that characterize the wave in the plasma. So that the parameters that characterize the plasma from the electromagnetic viewpoint (as the dielectric permittivity ε_{ik}, the susceptibility χ_{ik}, the

conductivity σ_{ik} and so on) will depend on these parameters ($\underline{E}$ and ω mainly) and on the parameters that characterize the plasma from the kinetic viewpoint (as the mean free path λ, the collision frequency ν and so on). In this way, in the relations that give the electric polarizability $\underline{P}$ or the conduction current density $\underline{j}_c$ that is;

$$P_i = \chi_{ik} E_k$$

$$j_{c_i} = \sigma_{ik} E_k$$

the electrical parameters ε_{ik}, χ_{ik}, σ_{ik} will depend on $\underline{E}$ also and the quantities ε_{ik} and σ_{ik} will no longer be proportional to $\underline{E}$. Therefore, the electrodynamic processes in the plasma and the same propagation of electromagnetic waves will become non-linear.

There are two types of approach to the theory of non-linear effects in plasma: the first one is a generalization of the mean free path method and the second one is the statistical method based on Boltzmann equation.

The fundamental equation of the first method is Langevin's equation

$$\frac{d\underline{v}}{dt} + \nu \underline{v} = \frac{e}{m} \left[\underline{E} + \frac{\underline{v}}{c} \times \underline{\mathcal{H}} \right] \qquad (1)$$

e, m, ν, $\underline{v}$ being respectively charge, mass, collision frequency and velocity of the electron; $\underline{E}$ is the electric field of the radiowave, $\underline{\mathcal{H}}$ the external magnetic field in which the plasma is immersed. Knowing $\underline{E}$ eq. (1) allows us to calculate $\underline{v}$, furthermore putting by definition :

$$\underline{j}_t = n e \underline{v}$$

where n is the electronic density and $\underline{j}_t$ is the total current density, it is possible to calculate the conductivity σ_{ik} and the dielectric susceptibility χ_{ik} by means of the equation

$$\underline{j}_t = \underline{\underline{\sigma}} \cdot \underline{E} + \frac{\partial \underline{P}}{\partial t}$$

In this way are obtained the expressions of the dielectric permittivity ε_{ik} ($\varepsilon_{ik} = 4\pi [\chi_{ik} + \delta_{ik}]$) and of the conductivity σ_{ik} that must be substituted in the equations of the wave propagation. It is easy to see that these expressions of ε_{ik} and of σ_{ik} do not depend on $\underline{E}$ and so the propagation, also of a very strong wave, is linear. To overcome this difficulty [1] the electronic collision frequency ν (that enters into the expressions of ε_{ik} and of σ_{ik}) is considered dependent on the electronic temperature T_e e.g. in the following way*

$$\nu(T_e) = \nu^{(o)} \sqrt{\frac{T_e}{T}} \tag{2}$$

being T the molecular temperature and $\nu^{(o)}$ the collision frequency in the absence of an electric field ($T_e = T$). In order to calculate T_e a differential equation is established equating dT_e/dt to the net power gained by the electron, in the following way

$$\frac{dT_e}{dt} = \frac{2}{3\ kn} \underline{j}_t \cdot \underline{E} - \delta(T_e - T)\ \nu(T_e), \tag{3}$$

Given the rather complicated structure of this equation, the usual method to solve it consists in disregarding, at first , the dependence of ν on T_e. In the final formulas obtained in this way is introduced ν , given by (2), and so it is possible to calculate T_e as a function of the various parameters of the plasma among which is the electric field. In this way ε_{ik} and σ_{ik} depend on $\underline{E}$ and the equations for the propagation

---

* This dependence is valid for electron-molecule collisions, see ref. [1] .

become non-linear. It is clear that this type of approach to the theory of non linear propagation presents defects and contradictions that are typical of the mean free path method. This method, worked out particularly by Townsend and Huxley, is based on the calculation of the distance travelled by the electron under the influence of the accelerating field during the interval of time between two successive collisions. The mean distance travelled by an ensemble of electrons per unit time gives the diffusion velocity of the electrons through the gas. The intensity of the field is always supposed to be weak: the calculation is done by supposing that all the electrons have the same velocity, then the mean value of the results is calculated by means of an electronic distribution function, previously unknown. A maxwellian distribution function is employed to obtain results which do not contain the above mentioned mean values. This method has the drawback of being valid only for weak fields, which are not able to deviate the electronic distribution function from its stationary form (maxwellian), without the possibility of exactly defining the general validity and the order of magnitude of the employed approximations. Furthermore, according to Huxley [2] the method of the mean free paths can be employed whenever we suppose that among all the various electron-molecule types of collision the binary ones are by far the most important and that the motion of the particles can be divided into short periods, in which the collisions occur, separated by comparatively long intervals during which the interactions with the other particles can be neglected in comparison to the action of the field.

It is evident that if the behaviour of a physical system can be outlined in the above mentioned way, it can be analyzed by

means of Boltzmann's integrodifferential equation.
From a more general viewpoint, we do not see how it would be possibile to base the theory of non-linear propagation on the equation (1) of dynamics, when we are faced with an electron gas, with a statistical velocity distribution.

1. - The Fundamental Equations of the Statistical Method.

We are interested in the lower E layer of the ionosphere, i.e., the zone between 80 and 95 Km height.
The molecular temperature of this layer varies slowly from 205 °K (at 80 km) to 217 °K (at 95 km). Let us suppose that the temperature gradient is zero and let us take a mean molecular temperature equal to 210 °K. The molecular concentration is of about 10^{14} mol/cm^3 in this layer. The electron density varies from $5\cdot10^2$ el/cm^3 (at 80 km) to $5\cdot10^4$ el/cm^3 (at 95 km) and therefore it is much inferior to the molecular one.
The statistical state of the ionospheric plasma is described by means of two distribution functions: one $f(\underline{r}, \underline{v}, t)$ for the electrons, the other $F(\underline{R}, \underline{V}, t)$ for the molecules; these are the solutions of the following system:

$$\frac{\partial f}{\partial t} + \underline{v}\cdot\underline{\nabla}_{\underline{r}} f + \underline{\gamma}\cdot\underline{\nabla}_{\underline{v}} f = C_{11} + C_{12}$$

$$\frac{\partial F}{\partial t} + \underline{V}\cdot\underline{\nabla}_{\underline{R}} F + \underline{\Gamma}\cdot\underline{\nabla}_{\underline{V}} F = C_{21} + C_{22}$$

in which C_{11} represents the collision term for electron-electron interaction , C_{22} the collision term for molecule-molecule interactions and last C_{12} and C_{21}, the collision terms for electron-molecule interactions and where we have indicated with $\underline{\gamma}$ and $\underline{\Gamma}$, the external accelerations acting on the electronic and molecular gases respectively. Taking into account

the characteristics of the ionospheric plasma, the second of the written equations can be simplified in the following way:

$$C_{22} = 0$$

which, after integration, gives the following (maxwellian) distribution function for the molecular velocities,

$$F = N \left(\frac{M}{2\pi kT}\right)^{\frac{3}{2}} \exp\left\{-\frac{M V^2}{2 kT}\right\}$$

where N is the molecular density.

Moreover, as N is much greater than n (electronic density), the first equation of the system can be simplified, neglecting C_{11} with respect to C_{12}; in this way we obtain :

$$\frac{\partial f}{\partial t} + \underline{v}\cdot\underline{\nabla}_{\underline{r}} f + \frac{e}{m}\left[\underline{E} + \frac{\underline{v}}{c} \times \underline{\mathcal{H}}\right]\cdot\underline{\nabla}_{\underline{v}} f = C_{12} \quad (4)$$

C_{12} is given by the following integral operator that we will indicate with $J\{f\}$

$$C_{12} \equiv J\{f\} = \iint \left[f(\underline{v}')F(\underline{V}') - f(\underline{v})F(\underline{V})\right] g\sigma(\vartheta\cdot g)\; dV d\Omega \quad (5)$$

where $g = \left|\underline{v} - \underline{V}\right|$ is the modulus of the relative velocity, $\sigma(\vartheta, g)$ is the differential cross-section, ϑ the scattering angle, $\underline{v}'$ and $\underline{V}'$ are the velocities of the electron and of the molecule before the collision (after the collision they become $\underline{v}$ and $\underline{V}$ respectively).

Taking into account the characteristics of the ionospheric plasma it is possibile to simplify eq. (4) [3].

We have already said that δ, the mean energy lost by an electron in a collision is much less than 1. In the case of elastic collisions

$$\delta = \frac{2m}{M} \simeq 3.4 \cdot 10^{-5}$$

in the case of inelastic collisions the energy lost by collision is noticeable, but because these collisions are much less frequent than the first ones, the mean value of δ is always much less than 1. For this reason, the r.m.s. velocity of an electron is much greater, also in strong fields, than the mean directed velocity, so that the symmetric portion, in $\underline{v}$, of the electronic distribution function is much greater than its antisymmetric portion. Moreover, as δ is small, the collision term (5) can be expressed in differential form; in this way eq. (4) becomes the Boltzmann-Fokker-Planck equation. For these reasons let us develop the electronic distribution function is spherical harmonics in velocity space, putting:

$$f(\underline{r}, \underline{v}, t) = f_o(\underline{r}, v, t) + \underline{a}_v \cdot \underline{f}_1(\underline{r}, v, t) + \chi(\underline{r}, \underline{v}, t) \qquad (6)$$

where $\underline{a}_v = \frac{\underline{v}}{|\underline{v}|}$. If we put this equation into (4) and integrate all over $d\Omega$ (differential of the solid angle in velocity space) we obtain :

$$\frac{\partial f_o}{\partial t} + \frac{v}{3} \underline{\nabla}_{\underline{r}} \underline{f}_1 + \frac{e}{3mv^2} \frac{\partial}{\partial v} \left\{ v^2 \underline{E} \cdot \underline{f}_1 \right\} - \frac{\delta}{2v^2} \frac{\partial}{\partial v} \left[\nu v^3 \left(1 + \frac{kT}{mv} \frac{\partial}{\partial v}\right) f_o \right] = 0 \qquad (7)$$

Putting again eq. (6), multiplied by $\underline{a}_v$ into (4) and integrating again all over the solid angle $d\Omega$ we have :

$$\frac{\partial \underline{f}_1}{\partial t} + v \underline{\nabla}_{\underline{r}} f_o + \frac{e}{m} \frac{\partial f_o}{\partial v} \underline{E} + \frac{e}{mc} \underline{\mathcal{H}} \times \underline{f}_1 + \nu \underline{f}_1 = 0 \qquad (8)$$

where we have put: [1] , [3] , [4] :

$$J\left\{f_o\right\} = \frac{\delta}{2v^2}\frac{\partial}{\partial v}\left[\nu v^3\left(1+\frac{kT}{mv}\frac{\partial}{\partial v}\right) f_o\right]$$

$$J\left\{\underline{f}_1\right\} = -\nu\underline{f}_1$$

In this way, starting from (4), with the approximation (6), we have obtained (7) and (8). To establish these equations we employed the properties of monodromy and orthogonality of f_o and $\underline{f}_1$ expressed by (6) and we supposed that the function χ and the functions that can be obtained by it with the application of the various operators of eq. (4) are very small in respect to the function f_o and of the corresponding ones, obtained by applying to it the various operators of (4).

It is easy to see that if the electronic density varies smoothly and if the variation of the electronic current along a mean free path λ is small with respect to the product of the electronic density and the velocity, that is if :

$$\lambda\left|\frac{\partial j}{\partial x}\right| \ll n\langle v\rangle$$

then

$$\chi \simeq \sqrt{\delta}\left|\underline{f}_1\right| \simeq \delta f_o$$

So that the function χ is small not only with respect to f_o (as we supposed to derive eqs. (7) and (8)) but also with respect to $\left|\underline{f}_1\right|$. It is clear that with the system composed of (7) and (8) it is not possible to determine corrections to f_o of the order of δf_o.

2. - The Electronic Distribution Function for a Monochromatic Wave. (*)

Let us suppose that the plasma is homogeneous, in this case eqs. (7) and (8) become:

$$\frac{\partial f_o}{\partial t} + \frac{e}{3mv^2}\frac{\partial}{\partial v}\left\{v^2 \underline{E}\cdot\underline{f}_1\right\} - \frac{\delta}{2v^2}\frac{\partial}{\partial v}\left\{\nu v^3\left[1+\frac{kT}{mv}\frac{\partial}{\partial v}\right] f_o\right\} = 0 \qquad (9)$$

$$\frac{\partial \underline{f}_1}{\partial t} + \frac{e}{m}\frac{\partial f_o}{\partial v}\underline{E} + \frac{e}{mc}\underline{\mathcal{H}}\times\underline{f}_1 + \nu\underline{f}_1 = 0. \qquad (10)$$

In order to solve this system of four equations we must define two relaxation times. The first is the relaxation time τ_E of the electric field that is defined as the time necessary to change substantially the field (if $\underline{E} = \underline{E}_o \cos \omega t$ then $\tau_E \simeq \omega^{-1}$). The second is the relaxation time τ_r of the electronic energy; τ_r is of the order of $(\delta\nu)^{-1}$ as can be seen by integrating (3) with $E_o = 0$. Following Gurevich [5] we can distinguish two cases: the first in which the electric field varies slowly ($\tau_E \gg \tau_r$), the second in which it varies quickly ($\tau_E \ll \tau_r$). Now we must consider the case in which the field varies quickly because we study the phenomena related with the propagation of a monochromatic wave :

$$\underline{E} = \underline{E}_o \cos \omega t \qquad (11)$$

which pulsation is equal, or very near, to the gyropulsation of the medium. In fact the value of the magnetic field of the earth, that enters into (10) varies, in the zone of the ionosphere in which we are interested, between 0.35 Gauss

(*) Some results of this section are already contained in the paper "Sull'eccitazione dell'"airglow" per mezzo di radio onde" of one of us (P.C.) published in Nuovo Cimento Suppl. vol. XIX Serie X N° 2 (1961).

and 0.45 Gauss so that, being $\omega \simeq \omega_H$, $\tau_E \simeq 1.4 \cdot 10^{-7}$ sec, because $\omega_H = e\,\mathcal{H}/mc$; moreover $\tau_r \simeq (\delta\nu)^{-1}$ is about $4 \cdot 10^{-3}$ sec, being $\delta \simeq 3.4 \cdot 10^{-5}$ and $\nu \simeq 8 \cdot 10^{6}$ coll/sec. From these calculations we see that in our case $(\frac{\tau_E}{\tau_r}) << 1$. Therefore let us develop f_o and $\underline{f}_1$ into a power series of the parameter $(\frac{\tau_E}{\tau_r})$ putting :

$$f_o = f_{o0} + f_{o1} + f_{o2} + \dots$$

$$\underline{f}_1 = \underline{f}_{10} + \underline{f}_{11} + \underline{f}_{12} + \dots$$

we find that in the zero order approximation we can neglect the variation of the distribution function due to collisions (of the order of f/τ_r) in respect to the first term of eq.(9) (because of the order of $\frac{\partial f}{\partial t} \simeq f/\tau_E$) so $\frac{\partial f_{oo}}{\partial t} = 0$. Therefore :

$$f_{oo} = f_{oo}(v)$$

that is, in the zero order approximation, the symmetric part of the distribution function is independent of time.

If we put into (9)

$$\underline{f}_{10} = - \underline{u} \frac{\partial f_{oo}}{\partial v}$$

we see that $\underline{u}$ satisfies Langevin's equation (1). This equation can be solved for $\underline{E}$ given by (11) and gives the following expression for the persistent part of $\underline{f}_{10}$

$$\underline{f}_{10} = \underline{A}(\omega) \frac{\partial f_{oo}}{\partial v} \cos\omega t + \underline{B}(\omega) \frac{\partial f_{oo}}{\partial v} \sin\omega t. \qquad (12)$$

The two vectors $\underline{A}(\omega)$ and $\underline{B}(\omega)$ are given by:

$$
\begin{cases}
\underline{A}(\omega) = -\dfrac{eE_o}{m}\dfrac{1}{\left[(\omega+\omega_H)^2+\nu^2\right]\left[(\omega-\omega_H)^2+\nu^2\right]} \\
\quad \left\{-\omega_H(\omega_H^2-\omega^2+\nu^2)\dfrac{\underline{\mathcal{H}}\times\underline{E}_o}{\mathcal{H}E_o}+\nu(\omega_H^2+\omega^2+\nu^2)\dfrac{\underline{E}_o}{E_o}+\right. \\
\qquad \left.+\nu(\omega_H^2-3\,\omega^2+\nu^2)\dfrac{\omega_H^2\cos\Phi}{\omega^2+\nu^2}\dfrac{\underline{\mathcal{H}}}{\mathcal{H}}\right\} \\
\underline{B}(\omega) = -\dfrac{eE_o}{m}\dfrac{1}{\left[(\omega+\omega_H)^2+\nu^2\right]\left[(\omega-\omega_H)^2+\nu^2\right]} \\
\quad \left\{-2\,\omega\omega_H\nu\dfrac{\underline{\mathcal{H}}\times\underline{E}_o}{\mathcal{H}E_o}-\omega(\omega_H^2-\omega^2-\nu^2)\dfrac{\underline{E}_o}{E_o}+\right. \\
\qquad \left.+\omega(\omega_H^2-\omega^2+3\nu^2)\dfrac{\omega_H^2\cos^2\Phi}{\omega^2+\nu^2}\dfrac{\underline{\mathcal{H}}}{\mathcal{H}}\right\}
\end{cases}
\tag{13}
$$

where Φ is the angle between $\underline{E}_o$ and $\underline{\mathcal{H}}$.

Given that $f_o(v,t) \simeq f_{oo}(v) + f_{01}(v,t)$ and $f_{oo}(v) \gg f_{01}(v,t)$ we have :

$$
\frac{\partial f_{01}(v,t)}{\partial t} = \frac{\delta}{2v^2}\frac{\partial}{\partial v}\left[\nu v^3\left(1+\frac{kT}{mv}\frac{\partial}{\partial v}\right)f_o-\frac{2\,e}{3m\delta}\,v^2\,\underline{E}\cdot\underline{f}_{10}\right] = \frac{\delta}{2v^2}\frac{\partial\mathcal{J}}{\partial v}
$$

from which :

$$
\Delta f_{01}(v,t) = \left\{\frac{\delta}{2v^2}\frac{\partial\mathcal{J}}{\partial v}\right\}t
$$

which is not limited for $t \to +\infty$. Stipulating then the existence of the electronic distribution function we must impose the condition $\mathcal{J} = 0$, that is :

$$
\nu v\left(1+\frac{kT}{mv}\frac{\partial}{\partial v}\right)f_{oo}-\frac{2e}{3m\delta}\,\underline{E}\cdot\underline{f}_{10} = 0\,.
$$

This is the differential equation for f_{oo}. Seemingly it is not selfconsistent because the time is explicitly contained in the expression $\underline{E} \cdot f_{10}$, but noting that f_{oo} must contain only the r.m.s. value of the electric field, we obtain :

$$- \frac{2e}{3m\,\delta} \langle \underline{E} \cdot f_{10} \rangle_{\tau_E} = \nu \frac{e^2 E_o^2}{3m^2\,\delta} \varphi(\nu) \frac{df_{oo}}{dv}$$

having put

$$\varphi(\nu) = \frac{\cos^2 \Phi}{\omega^2 + \nu^2} + \frac{\sin^2 \Phi}{2} \left\{ \frac{1}{(\omega+\omega_H)^2 + \nu^2} + \frac{1}{(\omega-\omega_H)^2 + \nu^2} \right\} \qquad (14)$$

so that :

$$f_{oo} = C \exp \left\{ - \int_o^v \frac{mv\,dv}{kT + \frac{e^2 E_o^2}{3m\,\delta} \varphi(\nu)} \right\} \qquad (15)$$

where the constant C must be determined from the normalization condition :

$$\int_o^{\infty} 4\pi v^2 f_{oo}(v)\,dv = 1 \,.$$

We see that the method suggested by Gurevich is particularly clear and precise, because the distribution function in the n^{th} approximation is obtained by imposing the condition of boundedness for $t \longrightarrow +\infty$ of the subsequent approximation. Keeping in mind that the magnetic field $\underline{\mathcal{H}}$ produces a plasma anisotropy breaking up the plane polarized electromagnetic wave into two elliptically polarized waves, it is important to get the electronic distribution function for an elliptically polarized electric field $\underline{E}$. We express $\underline{E}$ in terms of its components along the three principal polarization axes :

$$\underline{E} = \underline{E}_{\parallel o} \cos \omega t + \underline{E}^+_{\perp o} + e^{i\,\omega t} + \underline{E}^-_{\perp o}\, e^{-i\,\omega t} \,.$$

Where $\underline{E}_{\parallel o}$ is the plane-polarized field parallel to $\mathcal{H}$ and $\underline{E}^{-}_{\perp o}$ and $\underline{E}^{+}_{\perp o}$ are two circularly-polarized fields in a plane perpendicular to $\mathcal{H}$ and rotating, respectively, in the same sense or in the opposite sense of the electrons in the magnetic field. For such an electric field the distribution function (15) is still valid provided that $E_o^2\varphi(\nu)$ is replaced by:

$$\underline{E}_o^2\varphi(\nu) \longrightarrow \frac{E^2_{\parallel o}}{\omega^2+\nu^2} + \frac{2\,E^{+2}_{\perp o}}{(\omega+\omega_H)^2+\nu^2} + \frac{2\,E^{-2}_{\perp o}}{(\omega-\omega_H)^2+\nu^2} \tag{16}$$

It is clear that at the gyromagnetic resonance ($\omega \simeq \omega_H$) the energy transferred to the plasma is mainly due to the action of the $E^{-}_{\perp o}$ wave. Therefore let us consider in $E_o^2\varphi(\nu)$ only the part due to this wave, in this case eq. (15) becomes:

$$\begin{cases} f_{oo}(v)=C_{oo}\ \exp\left\{-\displaystyle\int_0^v \frac{mv\ dv}{kT + \dfrac{2\ e^2\ E^{-2}_{\perp o}}{3m\ \delta}\ \dfrac{1}{(\omega-\omega_H)^2+\ v^2\ \lambda^{-2}}}\right\} = \\[2ex] =C_{oo}\left[1+\dfrac{v^2}{\left[(\omega-\omega_H)^2+\dfrac{2e^2E^{-2}_{\perp o}}{3m\,\delta\ kT}\right]\lambda^2}\right]^{\dfrac{e^2E^{-2}_{\perp o}\ \lambda^2}{3\delta k^2T^2}} \exp\left\{-\dfrac{mv^2}{2kT}\right\} \end{cases} \tag{17}$$

where we have put:

$$\nu(v) = \frac{v}{\lambda} \tag{18}$$

with $\lambda = (\pi a_o N)^{-1}$ being a_o the "radius" of the molecule, N the molecular density and λ the mean free path of the electrons that we take as a constant. According to some authors (see refs. [6] and [7] it would be better to choose ν propor-

tional to v^2. We have chosen a linear dependence for the greater simplicity of the final formulas obtained, this is tantamount to substituting in them a suitable value λ_{eff}, for the mean free path, that takes into account the dependence of λ on v Putting:

$$\gamma = \frac{e^2 \, E_{\perp o}^{-2} \, \lambda^2}{3 \, \delta \, k^2 \, T^2} \quad ; \qquad \mu = \frac{m \, \lambda^2}{2 \, kT} (\omega - \omega_H)^2 .$$

By means of (17) we can calculate the mean energy $<\varepsilon>$ absorbed by the electrons of the plasma under the action of the component $E_{\perp o}^-$ of the electric field, we get:

$$<\varepsilon> = \frac{3}{2} \, kT(\gamma+\mu)^{\frac{1}{2}} \; \frac{W \, (\frac{\gamma}{2} - \frac{3}{4}, \frac{\gamma}{2} + \frac{5}{4} \; ; \gamma + \mu)}{W \, (\frac{\gamma}{2} - \frac{1}{4}, \frac{\gamma}{2} + \frac{3}{4} \; ; \gamma + \mu)} \tag{19}$$

where W(k,m;z) is the Whittaker function [8] of parameters k, m and argument z. From (19) we have :

$$\left(\frac{\partial <\varepsilon>}{\partial \omega}\right)_{\omega=\omega_H} = 0 \tag{19.1}$$

So that the energy absorption is maximal at the perfect resonance, $\omega = \omega_H$ (See Figs. N° 1,2). From the asymptotic representation of the Whittaker functions, it is possible to see that

$$\lim_{\omega \to \infty} <\varepsilon(\omega)> = \lim_{\omega_H \to \infty} <\varepsilon(\omega_H)> = \frac{3}{2} \, kT$$

From which we see that if the frequency of the wave is far from the gyrofrequency, the electronic distribution function tends to become maxwellian with a temperature equal to the molecular one.

At the perfect resonance eq. (17) becomes :

$$f_{oo}^{(r)}(v) = C_{oo}^{(r)} \exp \left\{ - \int_o^v \frac{mv\, dv}{kT + \dfrac{2\, e^2 \bar{E}_{\perp o}^{2}\, \lambda^2}{3m\, \delta\, v^2}} \right\} \tag{20}$$

If the electronic temperature is much greater than the molecular one, that is if

$$\frac{3}{2}\, kT << \frac{e^2\, \bar{E}_{\perp o}^{2}\, \lambda^2}{m\delta\, v^2}$$

eq (20) becomes:

$$f_{oo}^{*(r)}(v) = C_{oo}^{*(r)} \exp \left\{ - \frac{3}{8}\, \delta \left(\frac{m}{e\, \bar{E}_{\perp o}\, \lambda} \right)^2 v^4 \right\} \tag{21}$$

We have Druyvesteyn's distribution function [9] wherein appears the r.m.s. value of the electric field as it should be. With the aid of (21) we can calculate the fraction $P^{*(r)}(v \geqslant v_o)$ of electrons having a velocity exceeding a certain value v_o and the mean energy $<\varepsilon^{*(r)}>$ of the electrons. These quantities are given by :

$$P^{*(r)}(v \geqslant v_o) = \frac{W\left(-\frac{1}{8}\,,\ \frac{3}{8};\ \frac{3}{8}\, \delta \left(\frac{m\, v_o^2}{e\bar{E}_{\perp o}\, \lambda} \right)^2 \right)}{\Gamma\left(\frac{3}{4}\right) \left[\frac{3\delta}{8} \right]^{3/4} \left(\frac{m\, v_o^2}{e\bar{E}_{\perp o}\, \lambda} \right)^{1/4} \exp \left\{ + \frac{3}{16}\, \delta \left(\frac{mv_o^2}{e\bar{E}_{\perp o}\, \lambda} \right)^2 \right\}} \tag{22}$$

$$<\varepsilon^{*(r)}> = \frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} \sqrt{\frac{2}{3\,\delta}}\ e\, \bar{E}_{\perp o}\, \lambda \tag{23}$$

Measuring $<\varepsilon^{*(r)}>$ in eV, $\bar{E}_{\perp o}$ in Volt/cm, λ in cm and the pressure p in mmHg we have

$$<\varepsilon^{*(r)}> = 103.58\, \bar{E}_{\perp o}\, \lambda = 6.73\, \frac{\bar{E}_{\perp o}}{p} \tag{23.1}$$

We have found several expressions for the electronic distribution function in a slightly ionized gas under the action of an alternating electric field and a constant magnetic field, considering only elastic collision between electrons and gas molecules. As a result of such calculations (see fig. 3 and eq. (23)) it was found that for $\frac{E_{10}}{p} \simeq 1$ (that is, for a field intensity of some millivolt per cm and a pressure of some 10^{-3} mmHg), it is possible to have electrons accelerated in such a way that their mean energy is several eV so that they can collide inelastically with the molecules.

Let us now study the effect of these collisions on the electronic distribution function. In a number of researches (Davydov [3] , Druyvesteyn [9], Smit [10]), approximate methods have been developed in order to take into account this effect and recently a detailed study has been carried out by Kovrizhnykh [11]. The result achieved by him solves,in principal the problem raised by us, even if the final formulas are generally cumbersone to be numerically calculated. Due to its laboriousness and to the uncertainty of the experimental data required for its application, Kovrizhnykh's method is difficult to be employed for pratical purposes. Consequently, we thought it advisable to adopt, for the evaluation of the effect of inelastic collisions on the electronic velocity distribution, a semiempirical procedure particularly useful at least as far as our problem is concerned. Let us consider a gas (like air or the ionosphere) consisting essentially of poliatomic molecules. In this gas not only the levels of the electronic configurations but also the rotation and oscillation levels whose energy is rather low (of the order of $(10^{-4} \div 10^{-2})$ eV for rotation levels and $(0.1 \div 0.5)$ eV for oscillation levels) can be excited. Therefore inelastic

scattering with molecular excitation takes place also at low temperature and can absorb a considerable amount of electron energy. Keeping in mind the expression of the collision term for elastic collisions:

$$J^{(el)}\left\{f\right\} = + \frac{1}{2v^2}\frac{\partial}{\partial V}\left[v^2 \delta\nu \left(\frac{kT}{m}\frac{\partial f}{\partial V} + vf\right)\right]$$

we write down in the same way the corresponding term for inelastic collisions:

$$J^{(inel)}\left\{f\right\} = + \frac{1}{2v^2}\frac{\partial}{\partial V}\left[v^2 R(v) \left(\frac{kT}{m}\frac{\partial f}{\partial V} + vf\right)\right]$$

where $R(v)$ is, corresponding to $\delta\nu$, the electron energy loss due to inelastic collisions. We can then take into account this term, by substituting in the formulas for the energy distribution of the electrons for elastic collision only, the expression $\delta\nu$ with

$$R_{tot}(v) = \delta\nu(v) + R(v)$$

which corresponds to the introduction of an "efficient" energy loss for collisions given by :

$$\delta^{(eff)}(v) = \frac{R_{tot}(v)}{\nu^{(eff)}(v)},$$

The energy loss $\delta^{(eff)}(v)$ as a function of the electron velocity has been experimentally determined in many papers (for instance [12]). From the experimental data one can deduce that, up to energies of about 2 eV, for air and the ionosphere, $\delta^{(eff)}$ is nearly constant and in both cases it corresponds to about $1.6\cdot10^{-3}$ (to be compared with $\delta=3.4\cdot10^{-5}$ for elastic collisions only). Let us now note that in the formulas for electronic energy distribution in the case of

elastic collisions only, the parameter δ appears only through the ratio E^2/δ . This leads to the conclusion that, in order to take into account the effect of inelastic collisions, it is sufficient to substitute E^2/δ with $E^2/\delta^{(eff)}$ in the final formulas. Thus we can say that the effect of the inelastic collisions is to reduce the value of E^2 to a value $E^2_{(red)}$ such that $E^2_{(red)}/\delta = E^2/\delta^{(eff)}$. We have then

$$E_{(red)} = E\sqrt{\frac{\delta}{\delta^{(eff)}}}$$

For air and the ionosphere, we shall obtain

$$E_{(red)} = E\sqrt{\frac{3.4\cdot 10^{-5}}{1.6\cdot 10^{-3}}} \simeq \frac{1}{5} E$$

The electronic mean energy is therefore reduced (see Fig.No.4) from its value (23.1) for

$$\omega = \omega_H \qquad \frac{3}{2} kT << \frac{e^2 \; E^{-2}_{1o} \; \lambda^2}{m \, \delta \, v^2}$$

to

$$<\varepsilon^{**(r)}> = \frac{1}{5} <\varepsilon^{*(r)}> = 20.72 \; E^{-}_{1o} \lambda = 1.345 \; \frac{E^{-}_{1o}}{p} \qquad (23.2)$$

3. - <u>The Non Linear Dispersion Relation.</u> (*)

Taking (6) into account we have :

$$\underline{j}_t = \left[\underline{\underline{\sigma}} - i\,\omega \, \frac{\underline{\underline{\varepsilon}} - \underline{\underline{1}}}{4\pi} \right] \cdot \underline{E} = \frac{4\pi}{3} \, en \int_0^{\infty} v^3 \; \underline{f}_{10} \; dv$$

Putting the z axis along the direction of the external magnetic field $\underline{\mathfrak{H}}$ we have :

(*) A. Airoldi of our Institute has contributed to the work contained in this section. We take the occasion to thank her for the aid given in the numerical computations.

$$
\left\{
\begin{aligned}
&\sigma_{zz} = -\frac{\omega_p^2}{3}\int_0^\infty v^3 \frac{\nu}{\omega^2+\nu^2}\frac{\partial f_{oo}}{\partial v}\,dv \\
&\sigma_{xx}=\sigma_{yy} = -\frac{\omega_p^2}{3}\int_0^\infty v^3 \frac{1}{2}\left\{\frac{\nu}{(\omega+\omega_H)^2+\nu^2}+\frac{\nu}{(\omega-\omega_H)^2+\nu^2}\right\}\frac{\partial f_{oo}}{\partial v}\,dv \\
&\sigma_{xy} = -\sigma_{yx} = -\frac{\omega_p^2}{3}\int_0^\infty v^3 \frac{1}{2}\left\{\frac{\omega+\omega_H}{(\omega+\omega_H)^2+\nu^2}-\frac{\omega-\omega_H}{(\omega-\omega_H)^2+\nu^2}\right\}\frac{\partial f_{oo}}{\partial v}\,dv \\
&\sigma_{xz}=\sigma_{zx}=\sigma_{yz}=\sigma_{zy} = 0 \qquad \varepsilon_{xz}=\varepsilon_{zx}=\varepsilon_{yz}=\varepsilon_{zy} = 0 \\
&\frac{\varepsilon_{zz}-1}{4\pi} = \frac{\omega_p^2}{3}\int_0^\infty v^3 \frac{1}{\omega^2+\nu^2}\frac{\partial f_{oo}}{\partial v}\,dv \\
&\frac{\varepsilon_{xx}-1}{4\pi} = \frac{\varepsilon_{yy}-1}{4\pi} = \frac{\omega_p^2}{3}\int_0^\infty v^3 \frac{1}{2\omega}\left\{\frac{\omega+\omega_H}{(\omega+\omega_H)^2+\nu^2}+\frac{\omega-\omega_H}{(\omega-\omega_H)^2+\nu^2}\right\}\frac{\partial f_{oo}}{\partial v}\,dv \\
&\frac{\varepsilon_{xy}}{4\pi} = -\frac{\varepsilon_{yx}}{4\pi} = \frac{\omega_p^2}{3}\int_0^\infty v^3 \frac{1}{2\omega}\left\{\frac{\nu}{(\omega-\omega_H)^2+\nu^2}-\frac{\nu}{(\omega+\omega_H)^2+\nu^2}\right\}\frac{\partial f_{oo}}{\partial v}\,dv
\end{aligned}
\right. \tag{24}
$$

where ω_p is the plasma frequency, that is:

$$\omega_p^2 = \frac{4\pi e^2 n}{m}$$

The knowledge of the expressions of ε_{ik} and σ_{ik} allows us to study the problem of the propagation of electromagnetic waves in a plasma. In order to solve this problem it is possible to use in the linear approximation two methods that, in general, are equivalent. The first consists in the integration of the equations of the wave propagation, the second is related to the calculation and the study of the dispersion relation. The difference between these two methods lies in the fact that :

while the first one (the integration of the equations of the wave propagation) allows us to obtain quantitatively exact results only in relatively few cases, the second, on the contrary, gives a much more general description, also if it is only qualitative, of the propagation. More precisely we have in the case of a uniform plasma the two methods give equal results, in the non-uniform case the knowledge of the dispersion relation allows us, very often, to write down approximate solutions of the equations of the propagation. These solutions are in general the first term of a geometrical optics series. We must stress the fact that if we base the analysis of non linear propagation of a monochromatic wave launched in a plasma on eqs (24) we are compelled to admit that the deviations from linearity are not very strong. In fact we must suppose that there is no harmonic generation so that the main effect of non linear propagation consists in increasing the absorption coefficient and in modifying the velocity of the wave and not in a substantial variation of the shape of the monochromatic signal (caused by harmonic generation). For these reasons it ought to be possible to write down an approximate solution of the wave equation in the following way :

$$\underline{E} = \underline{E}_0 \exp\left\{i \frac{\omega}{c} (ct - \mathfrak{n} z)\right\} \qquad (25)$$

where now :

$$\mathfrak{n} = \mathfrak{n}(\underline{E}_0)$$

it is clear that for $E_0 \longrightarrow 0$ this formula must become the Appleton-Hartree formla.

We are now faced with the problem of calculating the integrals (24). If we consider the collision frequency ν a linear or a quadratic function of the velocity of the electron

it is easy to see that, in the case $\underline{\mathcal{H}} \neq \underline{0}$, it is not possible to give closed expressions for the components of $\underline{\underline{\varepsilon}}$ and $\underline{\underline{\sigma}}$. Furthermore, putting $\nu =$ const., it is possible to calculate all these integrals that are given by expressions that do not contain $\underline{E}_0$. On the other hand if we evaluate, in this case, the mean energy of an electron we easily arrive at the following formula

$$<\varepsilon> = \frac{3}{2}\, kT \left[1 + \frac{e^2\, E_0^2\, \varphi(\nu)}{3m\, \delta\, kT} \right]$$

so that, also if we have a noticeable heating of the plasma, the propagation is linear. All these difficulties can be overcome if we consider only weak fields for which the electronic distribution function does not appreciably deviate from its stationary form (maxwellian). In this way it is possible to calculate this function, developing f_{00} in Taylor series of E_0^2 , stopping the expansion after the first two terms and taking then $\nu =$ const. If we indicate the new distribution function obtained in this way with $f(v,E_0)$ we have

$$f(v,E_0) = f_{00}(v,E_0=0) + \left(\frac{\partial f_{00}}{\partial E_0^2}\right)_{E_0^2=0} E_0^2 \qquad (26)$$

that is:

$$f(v,E_0) = \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \left[1 + \frac{e^2 E_0^2 \varphi(\nu) v^2}{6\delta (kT)^2} \right] \exp\left\{ - \frac{mv^2}{2kT} \right\} \qquad (26.1)$$

the mean energy of an electron calculated by means of this function is given by :

$$<\varepsilon> = \frac{3}{2}\, kT \left[1 + \frac{5}{2}\, a \right] \qquad (27)$$

where :

$$a = \frac{e^2 E_o^2 \varphi(\nu)}{2m\,\delta\,kT} \tag{28}$$

so that we define as a weak field a field satisfying the following inequality (*) :

$$a \leqslant \frac{2}{5}\,' \tag{28.1}$$

By means of (26.1) it is possible to calculate all the integrals (24) and the following expressions are obtained:

$$\left\{\begin{array}{l}
\sigma_{zz} = \dfrac{1}{4\pi} A\,\omega_p^2 \dfrac{\nu}{\omega^2 + \nu^2} \\[2ex]
\sigma_{xx} = \sigma_{yy} = \dfrac{1}{4\pi} A \dfrac{\omega_p^2}{2} \left[\dfrac{\nu}{(\omega+\omega_H)^2 + \nu^2} + \dfrac{\nu}{(\omega-\omega_H)^2+\nu^2}\right] \\[2ex]
\sigma_{xy} = -\sigma_{yx} = \dfrac{1}{4\pi} A \dfrac{\omega_p^2}{2} \left[\dfrac{\omega+\omega_H}{(\omega+\omega_H)^2+\nu^2} - \dfrac{\omega-\omega_H}{(\omega-\omega_H)^2+\nu^2}\right] \\[2ex]
\sigma_{xz} = \sigma_{zx} = \sigma_{yz} = \sigma_{zy} = 0 \qquad \varepsilon_{xz} = \varepsilon_{zx} = \varepsilon_{yz} = \varepsilon_{zy} = 0 \\[2ex]
\quad \varepsilon_{zz} = 1 - A \dfrac{\omega_p^2}{\omega^2 + \nu^2} \\[2ex]
\varepsilon_{xz} = \varepsilon_{yy} = 1 - A \dfrac{\omega_p^2}{2\omega} \left[\dfrac{\omega+\omega_H}{(\omega+\omega_H)^2+\nu^2} + \dfrac{\omega-\omega_H}{(\omega-\omega_H)^2+\nu^2}\right] \\[2ex]
\varepsilon_{xy} = \varepsilon_{yx} = A \dfrac{\omega_p^2}{2\cdot\omega} \left[\dfrac{\nu}{(\omega+\omega_H)^2+\nu^2} - \dfrac{\nu}{(\omega-\omega_H)^2+\nu^2}\right]
\end{array}\right. \tag{24.1}$$

where:

$$A = 1 + a \tag{29}$$

(*) It must be said that this inequality can be too strong, what is important is that the second term at the right of (27) does not become much greater than the first.

for $E_0 \simeq 0$, that is $A = 1$, these formulas become those of the linear approximation.

Before calculating explicitly the dispersion relation it is better to find the expression of the components of $\underline{\underline{\varepsilon}}$ and $\underline{\underline{\sigma}}$ in the usual reference system for the study of wave propagation. To this aim let us put the external magnetic field $\underline{\mathcal{H}}$ in the plane y 0 z of the new reference system at an angle Θ with 0 z axis. It is easily seen that the matrix that allows us to pass from the old reference system to the new one is given by :

$$\gamma_i^m = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos\Theta & \sin\Theta \\ 0 & -\sin\Theta & \cos\Theta \end{vmatrix}$$

If we indicate with ε'_{ik} the components of the complex dielectric permittivity tensor (defined by $\varepsilon'_{ik}=\varepsilon_{ik}-i\,\frac{4\pi}{\omega}\,\sigma_{ik}$) in the new reference system and with $\tilde{\varepsilon}'_{ik}$ the components of the sale tensor in the old system, we have :

$$\varepsilon'_{ik} = \gamma_i^m\,\gamma_k^n\,\tilde{\varepsilon}'_{mn}$$

Introducing for simplicity's sake the notation

$$\Omega = \omega - i\nu \qquad\qquad \alpha = A\,\frac{\omega_p^2}{\omega\,\Omega\,(\Omega^2 - \omega_H^2)}$$

we have:

$$\varepsilon'_{xx} = 1 - \alpha\,\Omega^2$$

$$\varepsilon'_{xy} = -\,\varepsilon'_{yx} = i\alpha\,\Omega\,\omega_H\cos\Theta$$

$$\varepsilon'_{xz} = -\,\varepsilon'_{zx} = i\alpha\,\Omega\,\omega_H\sin\Theta$$

$$\varepsilon'_{yy} = 1 - \alpha\,(\Omega^2 - \omega_H^2\sin^2\Theta)$$

$$\varepsilon'_{yz} = \varepsilon'_{zy} = -\alpha\,\omega_H^2\sin\Theta\cos\Theta$$

$$\varepsilon'_{zz} = 1 - \alpha\,(\Omega^2 - \omega_H^2\cos^2\Theta)\,.$$

Let us consider the two Maxwell equations :

$$\underline{\nabla} \times \underline{H} = \frac{1}{c}\frac{\partial \underline{D}}{\partial t} + \frac{4\pi}{c}\underline{j}_c$$

$$\underline{\nabla} \times \underline{E} = -\frac{1}{c}\frac{\partial \underline{H}}{\partial t}$$

for the plane electromagnetic wave (25) propagating along the z axis they give:

$$\frac{\partial^2 \underline{E}}{\partial z^2} - \frac{\omega^2}{c^2}\underline{\underline{\varepsilon}}' \cdot \underline{E} = 0 \qquad (30)$$

the two Maxwell equations and eq. (30) are equivalent if we suppose that there is not harmonic generation of the fundamental frequency ω. Furthermore we have :

$$\frac{\partial \underline{E}}{\partial z} + i\frac{\omega}{c} n \underline{E} = -i\frac{\omega}{c} z \frac{\partial n}{\partial z}\underline{E}$$

$$\frac{\partial^2 \underline{E}}{\partial z^2} + \frac{\omega^2}{c^2} n^2 \underline{E} = -2\left[i\frac{\omega}{c} + \frac{\omega^2}{c^2} n z\right]\frac{\partial n}{\partial z}\underline{E} - \frac{\omega^2}{c^2} z^2 \left[\frac{\partial n}{\partial z}\right]^2 \underline{E} - i\frac{\omega}{c} z \frac{\partial^2 n}{\partial z^2}\underline{E}$$

but if the electric field is sufficiently weak the propagation will be almost linear so that it ought to be possible to disregard the right member of the last equations; this, on the other hand, agrees with the spirit of the perturbation technique that we have adopted (*). So :

$$\frac{\partial^2 \underline{E}}{\partial z^2} + \frac{\omega^2}{c^2} n^2 \underline{E} = 0$$

in this way eq. (30) becomes :

(*) This approximation has also been adopted by Ginzburg & Gurevich ref [1] second part, see particularly eqs. (3.5) and (3.6).

$$\left[\varepsilon'_{xx} - n^2\right] E_{ox} + \varepsilon'_{xy} E_{oy} + \varepsilon'_{xz} E_{oz} = 0$$

$$\varepsilon'_{yx} E_{ox} + \left[\varepsilon'_{yy} - n^2\right] E_{oy} + \varepsilon'_{yz} E_{oz} = 0$$

$$\varepsilon'_{zx} E_{ox} + \varepsilon'_{zy} E_{oy} + \varepsilon'_{zz} E_{oz} = 0$$

where, now, this is not a linear homogeneous system in the three unknowns E_{ox}, E_{oy}, E_{oz} as it happens in the linear case, but it is a more complicated system owing to the presence of E_o^2 in the expressions of ε'_{ik}. However, in this case too, it is easily shown that there exists a non zero solution for E_{ox}, E_{oy}, E_{oz} provided that the following determinant is equal to zero :

$$\begin{vmatrix} \varepsilon'_{xx} - n^2 & \varepsilon'_{xy} & \varepsilon'_{xz} \\ \varepsilon'_{yx} & \varepsilon'_{yy} - n^2 & \varepsilon'_{yz} \\ \varepsilon'_{zx} & \varepsilon'_{yz} & \varepsilon'_{zz} \end{vmatrix} = 0$$

from which we have (compare with eq. (6.1) in ref. [13]) :

$$n^2 = 1 - \frac{A\,X}{1 - iZ - \dfrac{Y \sin^2\Theta}{2\left[1-iZ-AX\right]} \pm \sqrt{\dfrac{Y^2 \sin^4\Theta}{4\left[1-iZ-AX\right]^2} + Y^2\cos^2\Theta}} \qquad (31)$$

being :

$$X = \frac{\omega_p^2}{\omega^2} \qquad Y = \frac{\omega_H}{\omega} \qquad Z = \frac{\nu}{\omega}$$

or (compare with eq. (2.3) in ref. [14])

$$n^2 = 1 - \frac{A\,b^2}{\sigma^2 - ir\sigma - \sigma\cos\Theta\left[B \pm \sqrt{1+B^2}\,\right]} \qquad (31.1)$$

being :

$$b = \frac{\omega_p}{\omega_H}, \qquad r = \frac{\nu}{\omega_H}, \qquad r_c = \frac{\sin^2 \Theta}{2 \cos \Theta}, \qquad \sigma = \frac{\omega}{\omega_H}$$

$$B = \frac{r_c \sigma \left(\sigma^2 - A\, b^2 + i\, r\, \sigma\right)}{\left(\sigma^2 - A\, b^2\right)^2 + r^2\, \sigma^2}$$

For A = 1 (linear case) eq. (31) becomes Appleton-Hartree's formula. In this case, as it is known [7], the results obtained by the statistical theory and by the mean free path theory must coincide.

The two signs that appear in eq. (31) show that in the non linear case too, the field is split up into two components whose propagation is never independent, given that under the square root there appears the total value of the electric field. We will call these two waves ordinary generalized (O.G.) wave and extraordinary generalized (E.G.) wave, identifying them with the corresponding ones of Appleton (O.A. and E.A. respectively), in the limit $E_o \to 0$ for every propagation condition.

Let us put

$$W = \frac{e^2\, E_o^2}{2m\, \delta\, kT\, \omega^2}$$

and A becomes

$$A = 1 + W \left[\frac{\cos^2 \Phi}{1+Z^2} + \frac{\sin^2 \Phi}{2} \left\{ \frac{1}{(1+Y)^2 + Z^2} + \frac{1}{(1-Y)^2 + Z^2} \right\} \right] \qquad (32)$$

It is interesting to notice that the electronic density n, that is X, does not appear in (32) so that all the results that are obtained for the Appleton-Hartree formula when is varied only n are valid also now if we subsitute X with A X.

The zeroes of n^2 are located at the points :

$$AX + iZ = 1 \qquad\qquad AX + iZ = 1 \pm Y$$

and one of the values of n^2 is infinite for :

$$AX = (1-iZ)\,\frac{(1-iZ)^2-Y^2}{(1-iZ)^2 - Y^2\cos^2\Theta}\,.$$

In this way we see that for A = 1 we obtain the well known results of the linear theory [13] . Furthermore the two values of n^2 are infinite for :

$$Y = \pm\,(-1 \pm iZ) \qquad \text{if } \sin\Theta \neq 0$$

$$Z = \pm\, i \qquad \text{if } \cos\Theta \neq 0$$

these poles are also branch points for n if $\Theta = \frac{\pi}{2}$.
It is easily seen from (15) and (26.1) that in these points we have $f_{oo}(v) \equiv 1$, $f(v,E_o) = \infty$, for these reasons all the integrals (24) diverge. Moreover $A = \infty$ and in this case the perturbation technique that we have used is no more valid.
Given that the dispersion relation (31) depends on ω in a much more complicated way than Appleton-Hartree's formula, we have studied it using an electronic computer. By means of (31) we have calculated the refraction index μ and the absorption index χ using the relation :

$$n^2 = (\mu - i\,\chi)^2 = M - iN$$

from which :

$$\mu = \left[\frac{1}{2}\left\{\sqrt{M^2 + N^2} + M\right\}\right]^{\frac{1}{2}}$$

$$\chi = \left[\frac{1}{2}\left\{\sqrt{M^2 + N^2} - M\right\}\right]^{\frac{1}{2}}$$

In the numerical calculations we have taken as a constant :

(1) the molecular temperature T, T = 200 °K

(2) the external magnetic field $\mathcal{H}$, = 0.42 Gauss for which $\omega_H = 7.388\cdot10^6$ puls/sec

(3) the electronic density n, $n = 10^4$ el/cm^3

(4) the mean energy lost by an electron in a collision, $\delta = 1.6 \cdot 10^{-3}$

(5) the direction of the propagation, we have put $\Theta = 0$ (longitudinal propagation) in this case we have, for symmetry, $\Phi = \frac{\pi}{2}$.

We have varied :

(1) The amplitude E_0 of the electric field

(2) Its pulsation around the chosen value of ω_H

(3) The electronic collision frequency ν.

The figures show the results that we have obtained.

From the figures N° 5, 6, 7 it is possible to see that the absorption index is in the non linear case greater than that calculated by means of Appleton-Hartree's formula, moreover, it increases with increasing electric field. For a field equal to, or less than, 10^{-4} Volt/cm we do not have appreciable deviations from the linearity. For low values of the collision frequency (Fig. N° 5, $\nu = 2.5 \cdot 10^6$ coll/sec) the absorption index of the O.G. wave increases very much for $\omega \simeq \omega_H$; with increasing ν this increase disappears (Figs. N° 6, 7) and for equal E_0 and $\frac{\omega}{\omega_H}$ the ratio between the values of χ for the O.G. wave (and E.G. wave) and the values of χ for the O.A. wave (and E.A. wave) tends to 1 with increasing ν.

As it is seen from Figs. N° 8, 9, 10 the behaviour of the refraction index μ is similar to that illustrated for χ when E_0, ω, ν are varied.

The fact that the propagation tends to become linear with increasing ν is easily understandable. In fact if E_0 and ω remain fixed the power transferred from the wave to the electrons is unaltered, while with increasing ν also the

power dissipated by the electrons increases due to the collisions, in this way the electronic temperature tends to become equal to the molecular one.

From the Figs. N° 8, 9 it can be seen that with increasing non linearity also the wave number of the O.G. wave increases. This fact generalizes a preceding result [15] according to which with increasing non linearity the wave number of an electromagnetic wave passing through an isotropic plasma is also increased.

The value of a given by (28) varies along the curves traced in the figures, for every curve the maximum value of this quantity is indicated that, in general, is achieved by a for $\omega \simeq \omega_H$. It is seen that for some propagation cases the condition (28.1) is not fulfilled. For this reason it is interesting to see if we obtain different results taking into account other terms in the development of f_{oo} in series of E_o^2. To this aim we have calculated the third term of the expansion that gives for A the value

$$A = 1 + a + \frac{a^2}{6} \tag{33}$$

and the fourth term for which

$$A = 1 + a + \frac{a^2}{6} - \frac{a^3}{54} \tag{34}$$

If we consider the method used to calculate (31) we see immediately that we obtain the new dispersion relations taking for A the values given by (33) and (34). The Figs. N° 11, 12, 13, 14 show the results obtained in this way. For electric fields equal to, or less than, 10^{-4} Volt/cm the results calculated by means of the values of A given by (29), (33), (34) are practically equal. For stronger electric fields, of the order of 10^{-3} Volt/cm, or for low collision frequencies

(Figs. N° 11, 13) the results are quite different. With increasing ν (Figs. N° 12, 14) these differences disappear, given that the non linearity of the medium is decreased for the reasons explained before.

The problem that arises when we want to treat strong fields and low values of the collision frequency does not consist in choosing a particular form for A but in showing that, in this case, we can still use the dispersion relation. In fact, as it can be seen from Fig. N° 8 it is possible to find some propagation conditions for which there is a strong difference between the phase velocities of the O.G. and E.G. waves. This causes, of course, a strong deformation of the monochromatic wave launched in the plasma. In this case we have harmonic generation and eqs (25), (30) loose their meaning. From this analysis we infer that it is not possible to treat, using a dispersion relation, the problem of non linear propagation of a rather strong electromagnetic wave in a plasma with a rather low value of the electronic collision frequency. We think that this problem could be solved in a satisfactory way by trying to integrate directly the system constituted by the equations of Boltzmann and of Maxwell. In fact, it is clear, that the electronic distribution function (15) from which we started in order to obtain eq. (31) is valid only if the wave remains monochromatic.

4. - The Electronic Distribution Function for an Amplitude Modulated Electric Field. (*)

The system of equations that we must integrate is given by eqs (9) and (10) that is :

$$\frac{\partial f_o}{\partial t} + \frac{e}{3mv^2}\frac{\partial}{\partial v}\left\{ v^2 \underline{E}\cdot\underline{f}_1 \right\} - \frac{\delta}{2v^2}\frac{\partial}{\partial v}\left[\nu v^3 \left(1+ \frac{kT}{mv}\frac{\partial}{\partial v}\right) f_o \right] = 0 \qquad (9)$$

(*) All the results contained in this section will appear, in a work by one of us (O.D.B.), in a more complete and elaborate form.

$$\frac{\partial \underline{f}_1}{\partial t} + \frac{e}{m} \frac{\partial f_o}{\partial v} \underline{E} + \frac{e}{mc} \underline{\mathcal{H}} \times \underline{f}_1 + \nu \underline{f}_1 = 0 \qquad (10)$$

where now

$$\underline{E} = \underline{E}_o \left[1 + \eta \cos(\alpha t + \beta) \right] \cos \omega t \qquad (35)$$

with $\omega_H \simeq \omega \gg \alpha$, $0 \leqslant \eta < 1$, $0 \leqslant \beta < 2\pi$.

We begin by defining three relaxation times: the first is the quick relaxation time of the electric field $\tau_{E.q.}$, it is determined by the frequency of the carrier and is of the order of ω^{-1}, that is $\tau_{E.q.} \simeq \omega^{-1}$. The second is the slow relaxation time of the electric field $\tau_{E.s.}$, it is determined by the frequency of the modulating signal, and is of the order of α^{-1}, that is $\tau_{E.s.} \simeq \alpha^{-1}$. The third is the relaxation time of the electronic energy τ_r that, as we know, is of the order of $(\delta \nu)^{-1}$.

We shall suppose that

$$\frac{1}{\omega_H} \simeq \tau_{E.q.} \ll \tau_r \ll \tau_{E.s.} \qquad (36)$$

If we remember the calculations of section 2 we see that (36) can be satisfied for values of α till to some kc/sec. From the fact that $\tau_{E.s.} \gg \tau_r$ derives that now the electronic distribution function must depend explicitly on time by means of the function $\cos(\alpha t + \beta)$. In this way we have the first term of (9) of the order of $f_o / \frac{1}{\alpha} \simeq \frac{1}{\tau_{E.s.}} \cdot f_o$. Let us put, as for the monochromatic wave (*) :

$$\underline{f}_1 = - \underline{u} \frac{\partial f_o}{\partial v} \cdot$$

(*) It is easy to see that this method of integration is tantamount to developing the functions f_o and $\underline{f}_1$ in a double series of powers of the parameters $\left(\frac{\tau_{E.q.}}{\tau_r}\right) \ll 1$ and $\left(\frac{\tau_{E.q.}}{\tau_{E.s.}}\right) \ll 1$. We shall calculate the first terms of these developments.

We have

$$\frac{\partial \underline{f}_1}{\partial t} = - \frac{\partial \underline{u}}{\partial t}\frac{\partial f_0}{\partial v} - \underline{u}\frac{\partial^2 f_0}{\partial t \partial v}$$

but $\underline{u}$ (which gives the current) must vary in time at the rate of the frequency ω of the carrier, so $\partial \underline{u}/\partial t \simeq \omega \underline{u}$, in this way, disregarding terms of the order of α/ω, we have:

$$\frac{\partial \underline{f}_1}{\partial t} = - \frac{\partial \underline{u}}{\partial t}\frac{\partial t_0}{\partial v}$$

and eq. (10) becomes the usual Langevin's equation :

$$\frac{\partial \underline{u}}{\partial t} + \nu \underline{u} = \frac{e}{m}\left[\underline{E} + \frac{\underline{u}}{c} \times \mathcal{H}\right]$$

We already know (see section 2) the solution of this equation for a monochromatic wave, taking into account that (35) can be written as:

$$\underline{E} = \underline{E}_0 \frac{1}{2}\eta \cos \left\{(\omega+\alpha)t+\beta\right\} + \underline{E}_0 \cos\omega t + \underline{E}_0 \frac{1}{2}\eta\cos\left\{(\omega-\alpha)t-\beta\right\}$$

and remembering eq. (13) we have the persistent part of $\underline{f}_1$ given by:

$$\begin{cases} f_1 = \Big[\underline{A}(\omega) + \frac{1}{2}\eta\left\{\underline{A}(\omega+\alpha)+\underline{A}(\omega-\alpha)\right\}\cos(\alpha t+\beta) + \\ \qquad + \frac{1}{2}\eta\left\{\underline{B}(\omega+\alpha)-\underline{B}(\omega-\alpha)\right\}\sin(\alpha t+\beta)\Big]\frac{\partial f_0}{\partial v}\cos\omega t + \\ \\ \Big[\underline{B}(\omega) + \frac{1}{2}\eta\left\{\underline{B}(\omega+\alpha)+\underline{B}(\omega-\alpha)\right\}\cos(\alpha t+\beta) + \\ \qquad + \frac{1}{2}\eta\left\{\underline{A}(\omega-\alpha)-\underline{A}(\omega+\alpha)\right\}\sin(\alpha t+\beta)\Big]\frac{\partial f_0}{\partial v}\sin\omega t . \end{cases} \tag{37}$$

In the calculation of $\underline{E}\cdot\underline{f}_1$ we disregard terms of the order of $\frac{\alpha}{\omega} = \frac{\tau_{E.q.}}{\tau_{E.s.}}$ that is, we put :

$$\begin{cases} \underline{E}_o \cdot \underline{A}(\omega-\alpha) = \underline{E}_o \cdot \underline{A}(\omega) = \underline{E}_o \cdot \underline{A}(\omega+\alpha) \\ \underline{E}_o \cdot \underline{B}(\omega-\alpha) = \underline{E}_o \cdot \underline{B}(\omega) = \underline{E}_o \cdot \underline{B}(\omega+\alpha) \end{cases} \tag{38}$$

and we define a function $\Xi(t)$ by :

$$\Xi(t) = \left[1+\eta \cos(\alpha t+\beta) \right]^2 \tag{39}$$

in this way eq (9) becomes :

$$\frac{\partial f_o}{\partial t} + \frac{e}{3mv^2} \frac{\partial}{\partial v} \left[v^2 \left\{ \underline{E}_o \cdot \underline{A}(\omega) \cos^2 \omega t + \underline{E}_o \cdot \underline{B}(w) \cos\omega t \sin\omega t \right\} \Xi(t) \frac{\partial f_o}{\partial v} \right] +$$

$$- \frac{\delta}{2\, v^2} \frac{\partial}{\partial v} \left[\nu v^3 \left(1+ \frac{kT}{mv} \frac{\partial}{\partial v}\right) f_o \right] = 0$$

On account of (36) we see that the first term of this equation (of the order of $f_o/\tau_{E.s.}$) can be disregarded with respect to the second term (of the order of $f_o/\tau_{E.q.}$) and to the third term (of the order of f_o/τ_r) so that taking the mean, over the period of the carrier, of the resulting equation we obtain :

$$\frac{\delta}{2v^2} \frac{\partial}{\partial v} \left[\nu v^3 \left\{ 1+\left(\frac{kT}{mv} + \frac{e^2 E_o^2}{3m^2\, \delta v} \varphi(\nu)\, \Xi(t)\right) \frac{\partial}{\partial v} \right\} f_o \right] = 0$$

a first integration gives

$$\nu v^3 f_o + \nu v^2 \left(kT + \frac{e^2 E_o^2}{3m\, \delta} \varphi(\nu)\, \Xi(t)\right) \frac{\partial f_o}{\partial v} = \Lambda(t)$$

where $\Lambda(t)$ is an unknown function of time, it can be easily seen that, unless it is not identically zero, f_o diverges strongly for $v = 0$. so that $\Lambda(t) \equiv 0$ and

$$f_o(v,t) = C(t) \exp \left\{ - \int_o^v \frac{mv\, dv}{kT + \frac{e^2 E_o^2}{3m\, \delta} \varphi(\nu)\, \Xi(t)} \right\} \tag{40}$$

The study of this distribution function is made easy by the fact that it is identical with (15) provided that we substitute E_o^2 with $E_o^2\, \Xi(t)$. In particular, we have the mean energy $<\varepsilon>$ depending on time, $<\varepsilon(t)>$, (*), and taking as before only the wave $E_{\perp o}^-$ it is given by (19) where now:

$$\gamma = \frac{e^2\, E_{\perp o}^{-2}\, \lambda^2}{3\, \delta\, k^2 T^2}\; \Xi(t)$$

furthermore we have, as before :

$$\left(\frac{\partial <\varepsilon\,(t)>}{\partial\, \omega}\right)_{\omega=\omega_H} = 0$$

Taking into account (38), eq (37) for $\underline{f}_1$ can be simplified in the following way

$$\underline{f}_1 = \Big[1+\eta\cos(\alpha t+\beta)\Big]\Big[\underline{A}(w)\cos\omega t+\underline{B}(w)\sin\omega t\Big]\frac{\partial f_o}{\partial\, v} \qquad (37.1)$$

furthermore putting :

$$\underline{j}_t = \frac{4\,\pi}{3}\, en\int_0^{\infty} v^3\, \underline{f}_1\, dv = \underline{\underline{\sigma}} \cdot \underline{E} + \frac{\partial\, \underline{P}}{\partial\, t}$$

it is possible to evaluate the conductivity and the dielectric permittivity of the plasma. It is easy to see that disregarding terms of the order of $\frac{\alpha}{\omega} = \frac{\tau_{E.\,q.}}{\tau_{E.s.}}$ the components of these tensors are given by eqs (24) provided that we substitute E_o^2 with $E_o^2\, \Xi(t)$.

We can still use approximation (26), so that the components of the complex dielectric permittivity tensor can be written as

$$\varepsilon'_{ik}(\omega,\alpha,t) = \varepsilon'_{ik}(\omega) + \Delta_{ik}(\omega,\alpha,t) \qquad (41)$$

(*) See Fig. N° 15.

being

$$\Delta_{ik}(\omega,\alpha,t) = \varepsilon'_{ik}(\omega)\,\frac{e^2 E_o^2\,\varphi(\nu)}{2m\,\delta\,kT}\,\Xi(t)$$

where $\varepsilon'_{ik}(\omega)$ is the expression that we obtain in the linear theory for the carrier. By eq (41) we see that ε'_{ik} is varied, around its steady value $\varepsilon'_{ik}(\omega)$, by the quantity $\Delta_{ik}(\omega,\alpha,t)$. Let us now launch in the plasma, besides the wave (35), also another wave of the type :

$$\underline{E}_\gamma = \underline{E}_{o\gamma}\cos(\gamma t + \Lambda) \tag{42}$$

Indicating with $\varphi(\omega,\nu)$ the function given by eq (14) and with $\varphi(\gamma,\nu)$ the same function with ω replaced by γ and where Φ is substituted with the angle between $\underline{E}_{o\gamma}$ and $\underline{\mathcal{H}}$, it is easily seen that the electronic distribution function $f_o(v,t)$ is either:

$$f_o(v,t)=C(t)\exp\left\{-\int_o^v \frac{mv\,dv}{kT+\frac{e^2}{3m\delta}\left[E_o^2\varphi(\omega,\nu)\Xi(t)+2E_{o\gamma}^2\varphi(\gamma,\nu)\cos^2(\gamma t+\Lambda)\right]}\right\} \tag{43a}$$

or

$$f_o(v,t)=C(t)\exp\left\{-\int_o^v \frac{mv\,dv}{kT+\frac{e^2}{3m\delta}\left[E_o^2\varphi(\omega,\nu)\Xi(t)+E_{o\gamma}^2\varphi(\gamma,\nu)\right]}\right\} \tag{43b}$$

Eq (43.a) is valid if $\tau_E \gg \tau_r$, (43.b) is valid if $\tau_E \ll \tau_r$

τ_E being the relaxation time of the electric field (42) which is: $\tau_E \simeq \gamma^{-1}$. Furthermore it must be noticed that in the calculation of eqs. (43 a,b) we have neglected terms proportional to the product of the two waves (35) and (42); it is possible

to make this approximation if there is no phase correlation between the two fields. Let us suppose now that the electric field (42) is small and that γ is far from the gyromagnetic frequency so that we can neglect in eqs (43) the part due to this wave, in this way $f_o(v,t)$ is given by eq (40). The equation for the propagation of the wave (42) is:

$$\underline{\nabla} \times \underline{\nabla} \times \underline{E}_\gamma + \frac{1}{c^2} \underline{\underline{\varepsilon}} \cdot \frac{\partial^2 \underline{E}_\gamma}{\partial t^2} + \frac{4\pi}{c^2} \frac{\partial \underline{j}_{cond}}{\partial t} = 0 \tag{44}$$

and given that the wave (35) contributes to $\underline{j}_{cond}$, the propagation conditions of the wave (42) will depend on those of wave (35). We can still use the approximation (26); an easy calculation shows that in this case eq. (44) becomes :

$$\underline{\nabla} \times \underline{\nabla} \times \underline{E}_\gamma + \frac{1}{c^2} \underline{\underline{\varepsilon}}(\gamma) \cdot \frac{\partial^2 \underline{E}_\gamma}{\partial t^2} + \frac{4\pi}{c^2} \underline{\underline{\sigma}}(\gamma) \cdot \frac{\partial \underline{E}_\gamma}{\partial t} + \mathfrak{J} = 0$$

where $\underline{\underline{\varepsilon}}(\gamma)$ and $\underline{\underline{\sigma}}(\gamma)$ are the expressions which we obtain in the linear theory for the wave (42) and where $\mathfrak{J}$ is given by :

$$\mathfrak{J} = \frac{4\pi}{c^2} \left[1 + \frac{e^2 E_o^2 \varphi(w, \nu)}{2m \delta kT} \Xi(t) \right] \underline{\underline{\sigma}}(\omega) \cdot \frac{\partial \underline{E}}{\partial t} .$$

It is clear that the wave (42) does not remain monochromatic and that these considerations constitute the starting point for a microscopic theory of cross-modulation.

Acknowledgements

This paper presents some results of a research program in progress at the Istituto di Scienze Fisiche dell'Università di Milano with the contribution of C.N.R.

REFERENCES

[1] V.L. Ginzburg, A.V. Gurevich - Soviet Physics. USPEKHI 3, 115, 175 (1960).

[2] L.G.H. Huxley - Phyl.Mag. 23, 210, 442 (1937); 29, 313(1940) Proc.Phys.Soc. 64B, 844 (1951)
see also:
L.G.H. Huxley, R.W. Crompton - The Motion of Slow Electrons in Gases - in "Atomic and Molecular Processe" ed. by D.R. Bates - Academic Press (1962).

[3] B.J. Davydov - Zurn.Eksp.Teor.Fiz. (U.S.S.R.) 6, 463(1936); 7, 1069 (1937).

[4] S. Chapman, T.G. Cowling - "The Mathematical Theory of Non-Uniform Gases" Cambridge U.P. (1960).

[5] A.V. Gurevich - Sov.Phys.JETP 3, 895 (1957).

[6] A.V. Phelps - Journ.of Applied Phys. 31, 1723 (1960)

[7] H.K.Sen, A.A. Wyller - Journ.of Geophysical Res. 65 3931 (1960).

[8] E.T. Whittaker, G.N. Watson - Modern Analysis - Cambridge U.P. (1958).

[9] M.J. Druyvesteyn, F.M. Penning - Rev.Mod.Phys. 12, 87(1940).

[10] J.A. Smit - Physica, 3, 543 (1936).

[11] L.M. Kovrizhnikh - Sov.Phys. JETP 10, 347 (1960)

[12] R.W. Crompton, L.G. Huxley, D.J. Sutton - Proc.Roy.Soc. 218A, 507 (1953).

[13] K.G. Budden - "Radio Waves in the Ionosphere", Cambridge U.P. (1961).

[14] R. Gallet - Whistlers in "Geophysics, the Earth's Environment" ed. by De Witt, Hieblot, Lebeau. Gordon & Breach (1963).

[15] M. Epstein - The Phys. of Fluids 5, 492 (1962)

List of Figures

Fig. N° 1 - Values of $<\varepsilon>$ calculated by means of eq.(19) in which: $\delta = 3.4\cdot10^{-5}$, $\omega_H = 7.388\cdot10^6$ puls/sec corresponding to $\mathcal{H} = 0.42$ Gauss, $\lambda = 10$ cm corresponding to a pressure $p = 6.7\cdot10^{-3}$ mmHg, T = 210 °K. It is possible to see clearly that the energy absorption maximal for $\omega = \omega_H$. Moreover for weak electric fields and for increasing $\frac{\omega}{\omega_H}$ all the curves tend to the same orizontal asymptote that is equal to the value of the energy of the electrons in thermal equilibrium with the molecules.

The various curves correspond to the following values of the $E^-_{\perp o}$ component of the electric field:

Curve N° 1	$E^-_{\perp o} = 10^{-3}$ Volt/cm
Curve N° 2	$E^-_{\perp o} = 7.5\cdot10^{-4}$ Volt/cm
Curve N° 3	$E^-_{\perp o} = 5\cdot10^{-4}$ Volt/cm
Curve N° 4	$E^-_{\perp o} = 2.5\cdot10^{-4}$ Volt/cm
Curve N° 5	$E^-_{\perp o} = 10^{-4}$ Volt/cm
Curve N° 6	$E^-_{\perp o} = 5\cdot10^{-5}$ Volt/cm
Curve N° 7	$E^-_{\perp o} = 10^{-5}$ Volt/cm

Fig. N° 2 - Values of $<\varepsilon>$ calculated by means of eq. (19) with the same values of the parameters as for Fig. N° 1. For this figure we can make the same observations as those that we made for Fig. N° 1. The various curves correspond to different values of $\frac{\omega}{\omega_H}$ namely :

Curve N° 1	$\omega = w_H$
Curve N° 2	$\omega = 0.7\ \omega_H$, $\omega = 1.3\ w_H$
Curve N° 3	$\omega = 0.5\ \omega_H$, $\omega = 1.5\ w_H$
Curve N° 4	$\omega = 2\ \omega_H$
Curve N° 5	$\omega = 3\ \omega_H$
Curve N° 6	$\omega = 4\ \omega_H$
Curva N° 7	$\omega = 5\ \omega_H$

Fig. N° 3 - Shape of the function $P^{*(r)}(v > v_o)$ given by eq. (22) for $\delta = 3.4 \cdot 10^{-5}$, $E^-_{\perp o} = 10^{-3}$ Volt/cm, $p = 10^{-3}$ mmHg (that is for $E^-_{\perp o}/p = 1$ corresponding, according to eq. (23.1), to a mean electronic energy equal to 6.73 eV), $\lambda = 66$ cm. From this figure it is possible to see that in this case we are compelled to take into account also of inelastic collisions.

Fig. N° 4 - Behaviour of $\langle \varepsilon \rangle$ (for $\omega = \omega_H$) as a function of $E^-_{\perp o}$

Curve N° 1 : values of $\langle \varepsilon \rangle$ calculated by means of eq. (19) in which $\delta = 3.4 \cdot 10^{-5}$

Curve N° 2 : values of $\langle \varepsilon \rangle$ calculated by means of eq.(23.1) in which $\delta = 3.4 \cdot 10^{-5}$

Curve N° 3 : values of $\langle \varepsilon \rangle$ calculated by means of eq. (19) in which $\delta = 1.6 \cdot 10^{-3}$

Curve N° 4 : values of $\langle \varepsilon \rangle$ calculated by means of eq. (23.2) in which $\delta = 1.6 \cdot 10^{-3}$

Fig. N° 5 - Values of χ for $\nu = 2.5 \cdot 10^6$ coll/sec

Curves N° 1 (O.G. and E.G.) $E_o = 10^{-3}$ Volt/cm $a_{1max} = 2.49$
Curves N° 2 (O.G. and E.G.) $E_o = 7.5 \cdot 10^{-4}$ " " $a_{2max} = 1.40$
Curves N° 3 (O.G. and E.G.) $E_o = 5 \cdot 10^{-4}$ " " $a_{3max} = 0.62$
Curves N° 4 (O.G. and E.G.) $E_o = 2.5 \cdot 10^{-4}$ " " $a_{4max} = 0.15$

Fig. N° 6 - Values of χ for $\nu = 5 \cdot 10^6$ coll/sec

Symbols same as in Fig. N° 5

$a_{1max} = 0.67$ $a_{2max} = 0.38$ $a_{3max} = 0.17$

Fig. N° 7 - Values of χ for $\nu = 7.5 \cdot 10^6$ coll/sec

Symbols same as in Fig. N° 5

$a_{1max} = 0.32$ $a_{2max} = 0.18$ $a_{3max} = 0.08$

Fig. N° 8 - Values of μ for $\nu = 2.5 \cdot 10^6$ coll/sec

Symbols same as in Fig. N° 5

$a_{1max} = 2.49$ $a_{2max} = 1.40$ $a_{3max} = 0.62$ $a_{4max} = 0.15$

Fig. N° 9 - Values of μ for $\nu = 5 \cdot 10^6$ coll/sec

Symbols same as in Fig. N° 5

$a_{1max} = 0.62$ $a_{2max} = 0.38$ $a_{3max} = 0.17$

Fig. N° 10 - Values of μ for $\nu = 7.5 \cdot 10^6$ coll/sec

Symbols same as in Fig. N° 5

$a_{1max} = 0.32$ $a_{2max} = 0.18$ $a_{3max} = 0.08$

Fig. N° 11 - Values of χ for $\nu = 2.5 \cdot 10^6$ coll/sec

1(O.G. and E.G.) a with A given by (29) $E_0 = 10^{-3}$ Volt/cm

1(O.G. and E.G.) a^2 with A given by (33) $E_0 = 10^{-3}$ Volt/cm

1(O.G. and E.G.) a^3 with A given by (34) $E_0 = 10^{-3}$ Volt/cm

3(O.G. and E.G.) a with A given by (29) $E_0 = 5 \cdot 10^{-4}$ Volt/cm

3(O.G. and E.G.) a^2 with A given by (33) $E_0 = 5 \cdot 10^{-4}$ Volt/cm

Fig. N° 12 - Values of χ for $\nu = 5 \cdot 10^6$ coll/sec

Symbols same as in Fig. N° 11.

Fig. N° 13 - Values of μ for $\nu = 2.5 \cdot 10^6$ coll/sec

Symbols same as in Fig. N° 11

Fig. N° 14 - Values of μ for $\nu = 5 \cdot 10^6$ coll/sec

Symbols same as in Fig. N° 11

Fig. N° 15 - Behaviour of $\langle \varepsilon(t) \rangle$ for $E^-_{1c} = 5 \cdot 10^{-4}$ Volt/cm, other constants same as in Fig. N° 1, for various values of η .


1.25
<ε>
in eV
1
0.75
0.50
0.25
1
2
3
4
5
6
7
0 0.3 0.7 1
0.5 0.9
2
3
4
5
$\frac{\omega}{\omega_H}$

Fig. N° 1

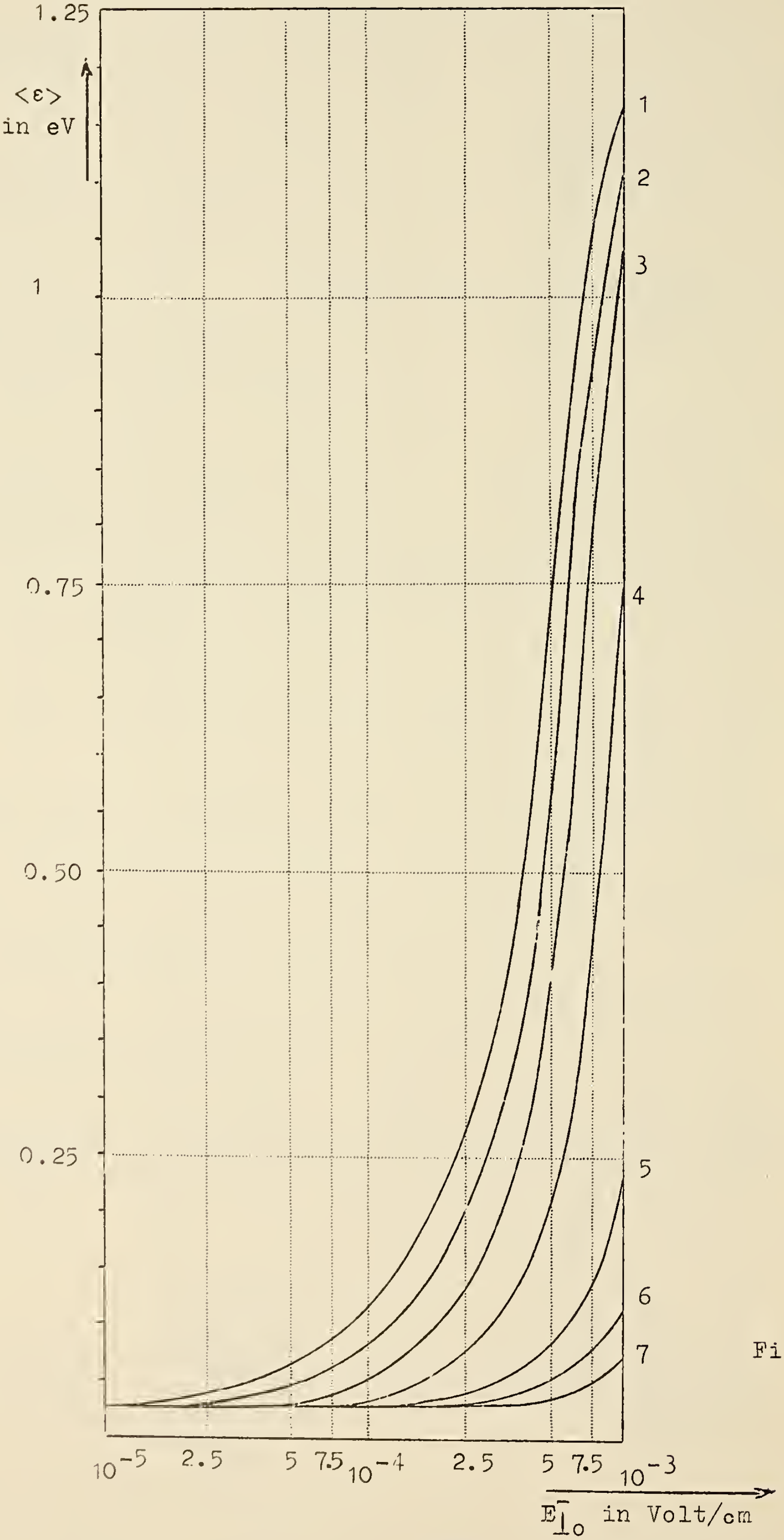

1.25
<ε>
in eV
1
0.75
0.50
0.25
1
2
3
4
5
6
7
10^-5 2.5 5 7.5 10^-4 2.5 5 7.5 10^-3
E⊥o in Volt/cm

Fig. N° 2

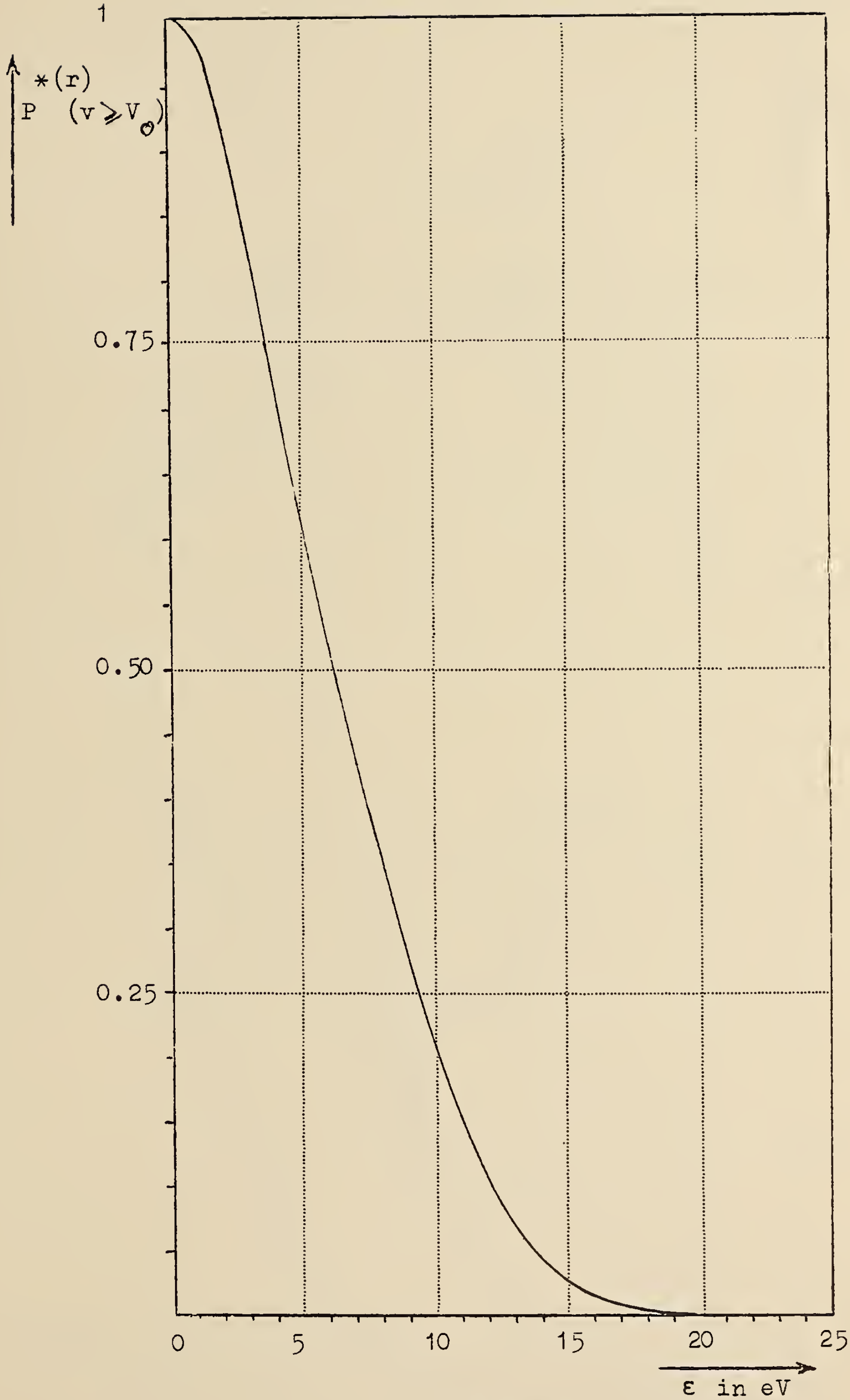


Fig. N° 3

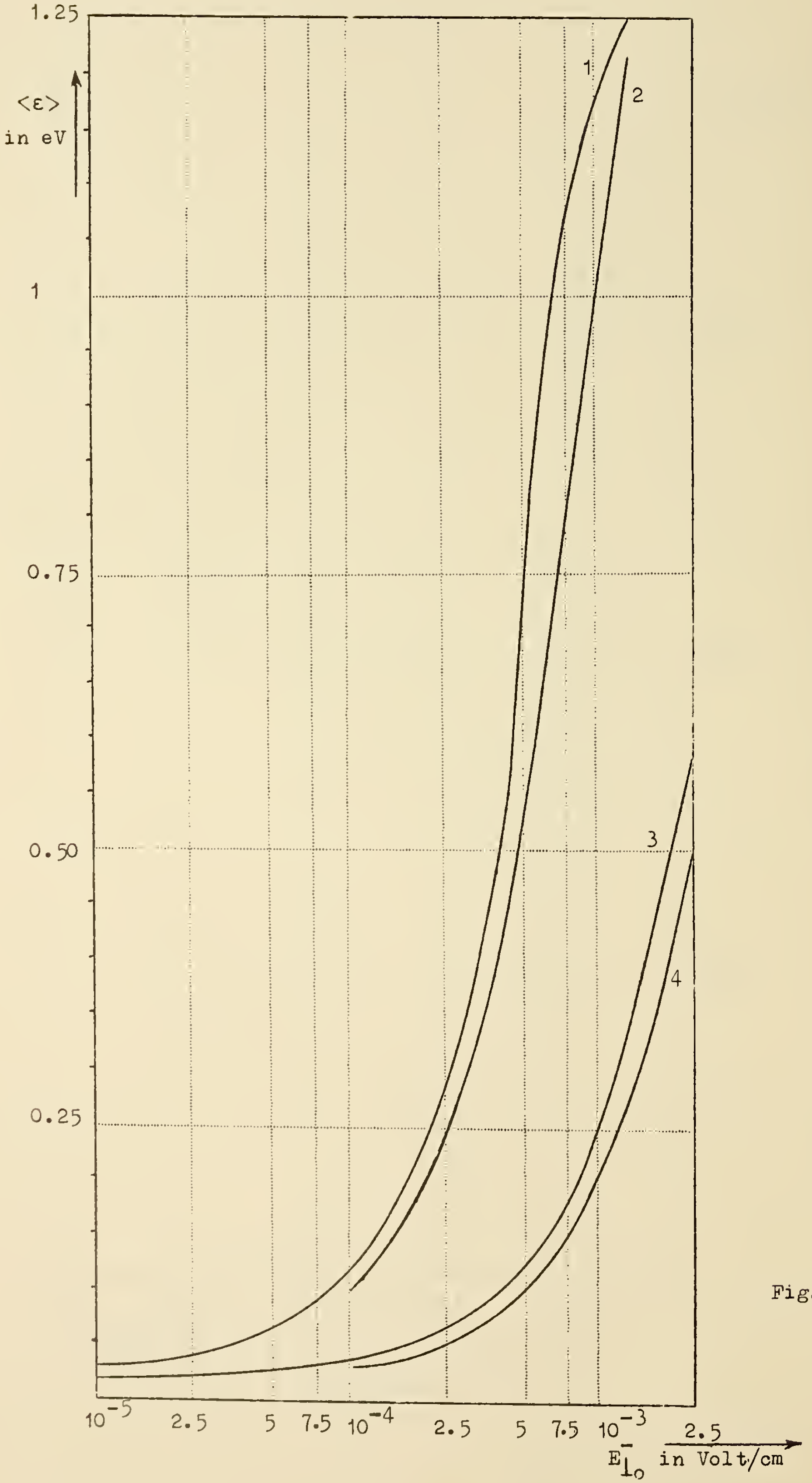
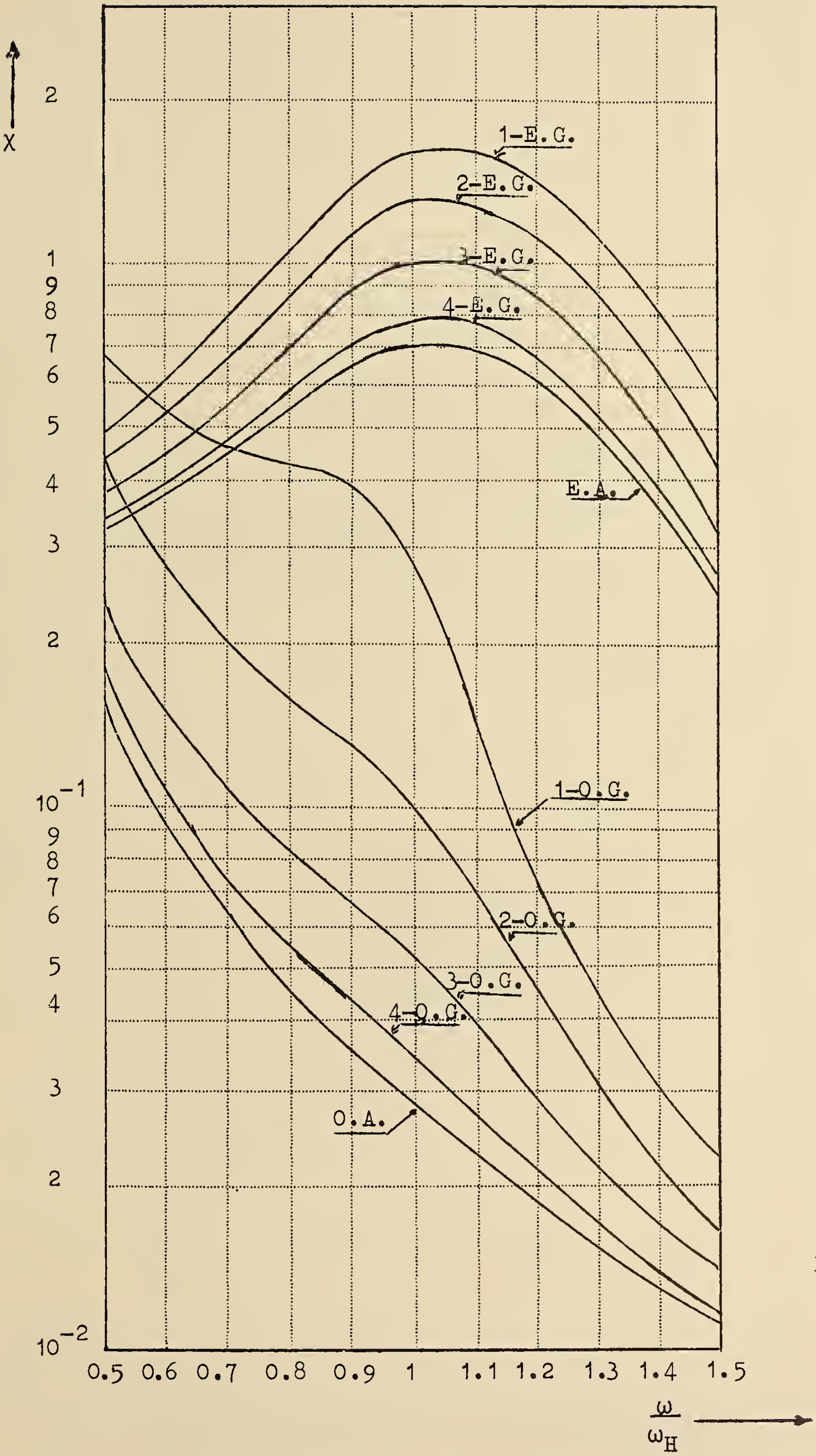
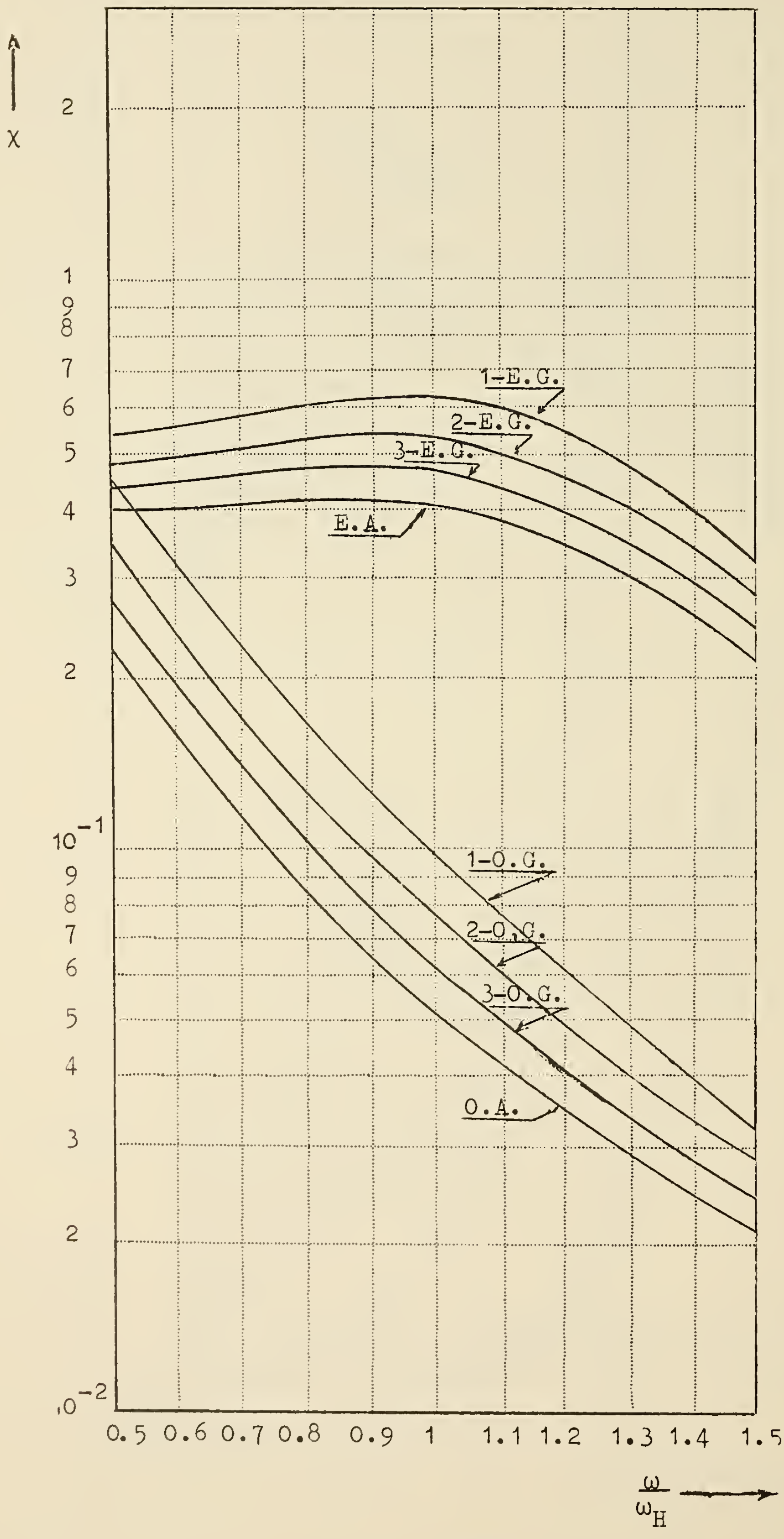

1.25
1
0.75
0.50
0.25
<ε>
in eV
1
2
3
4
10^{-5}
2.5
5
7.5
10^{-4}
2.5
5
7.5
10^{-3}
2.5
$\bar{E}_{1_0}$ in Volt/cm

Fig. N° 4



X
2
1
9
8
7
6
5
4
3
2
10^{-1}
9
8
7
6
5
4
3
2
10^{-2}
0.5
0.6
0.7
0.8
0.9
1
1.1
1.2
1.3
1.4
1.5
$\frac{\omega}{\omega_H}$
1-E.G.
2-E.G.
3-E.G.
4-E.G.
E.A.
1-O.G.
2-O.G.
3-O.G.
4-O.G.
O.A.

Fig. N° 5



χ
2
1
9
8
7
6
5
4
3
2
10^{-1}
9
8
7
6
5
4
3
2
10^{-2}
1-E.G.
2-E.G.
3-E.G.
E.A.
1-O.G.
2-O.G.
3-O.G.
O.A.
0.5 0.6 0.7 0.8 0.9 1 1.1 1.2 1.3 1.4 1.5
$\frac{\omega}{\omega_H}$

Fig. N° 6

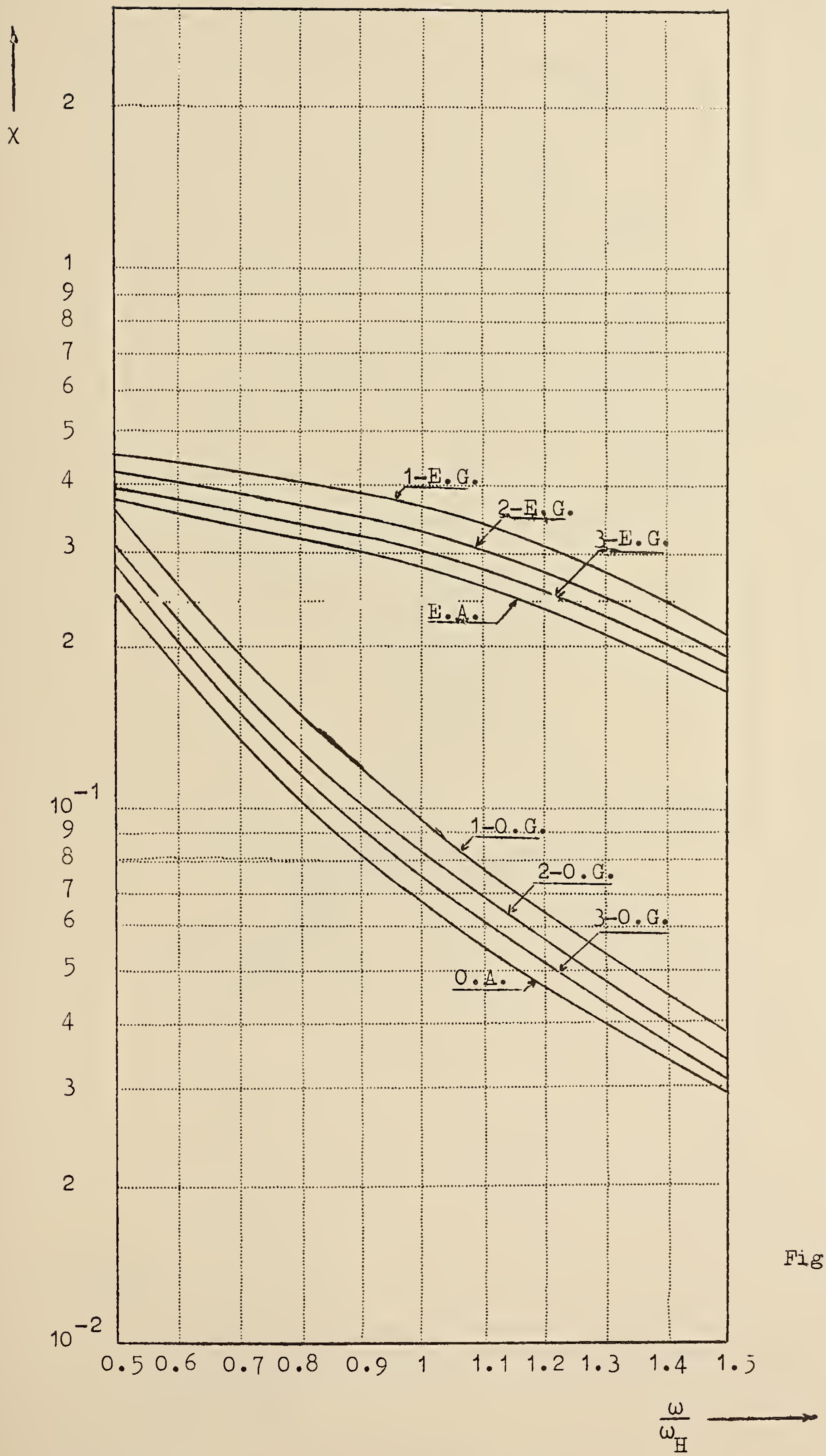
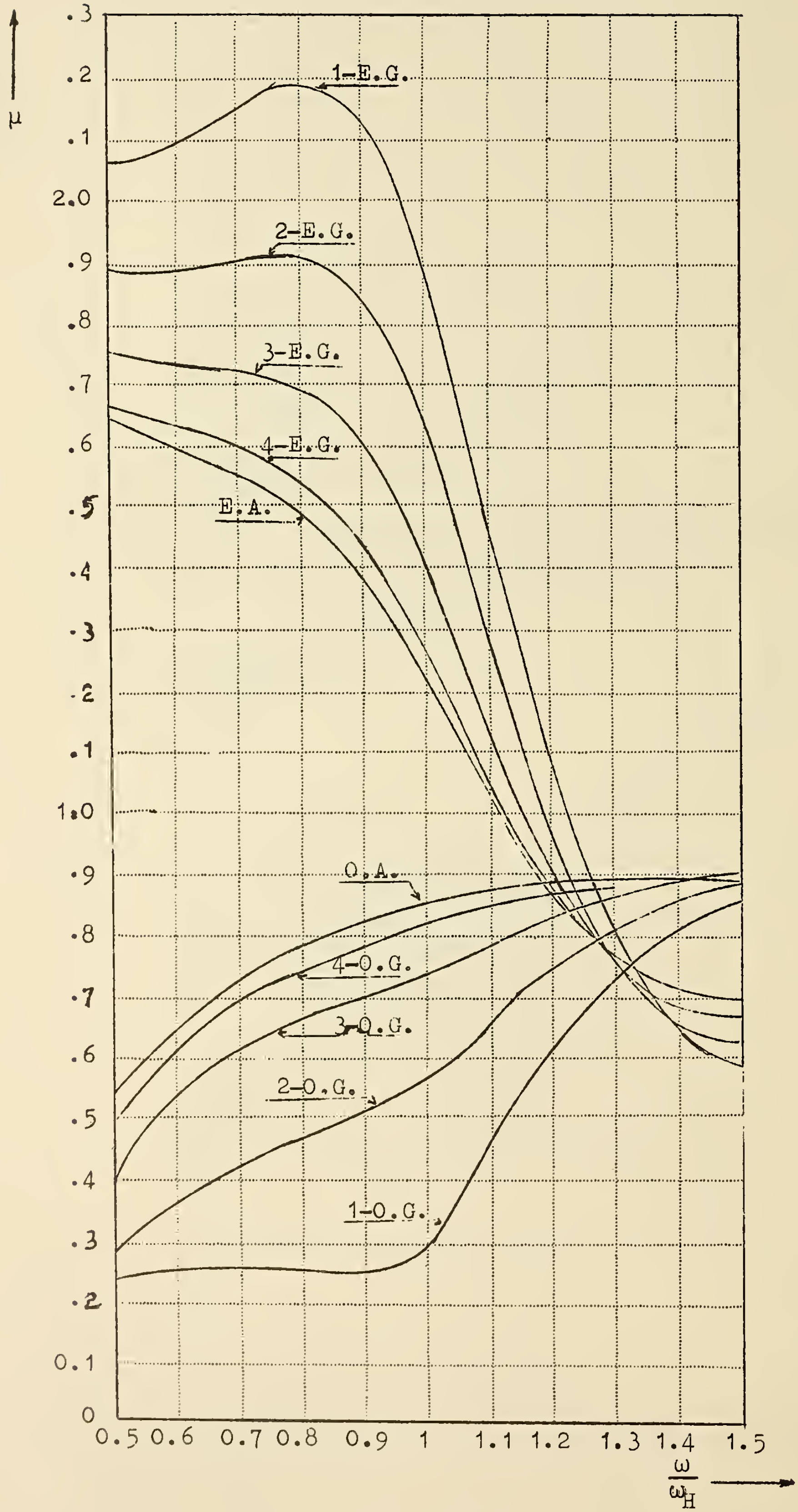

χ
2
1
9
8
7
6
5
4
3
2
10^{-1}
9
8
7
6
5
4
3
2
10^{-2}
1-E.G.
2-E.G.
3-E.G.
E.A.
1-O.G.
2-O.G.
3-O.G.
O.A.
0.5
0.6
0.7
0.8
0.9
1
1.1
1.2
1.3
1.4
1.5
$\frac{\omega}{\omega_H}$

Fig. Nº 7



μ
.3
.2
.1
2.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
1.0
.9
.8
.7
.6
.5
.4
.3
.2
0.1
0
1-E.G.
2-E.G.
3-E.G.
4-E.G.
E.A.
O.A.
4-O.G.
3-O.G.
2-O.G.
1-O.G.
0.5 0.6 0.7 0.8 0.9 1 1.1 1.2 1.3 1.4 1.5
$\frac{\omega}{\omega_H}$

Fig. N° 8


μ
.2
.1
2.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
1.0
.9
.8
.7
.6
.5
.4
.3
.2
0.1
0
1-E.G.
2-E.G.
3-E.G.
E.A.
O.A.
3-O.G.
2-O.G.
1-O.G.
0.5 0.6 0.7 0.8 0.9 1 1.1 1.2 1.3 1.4 1.5
$\frac{\omega}{\omega_H}$

Fig. N° 9

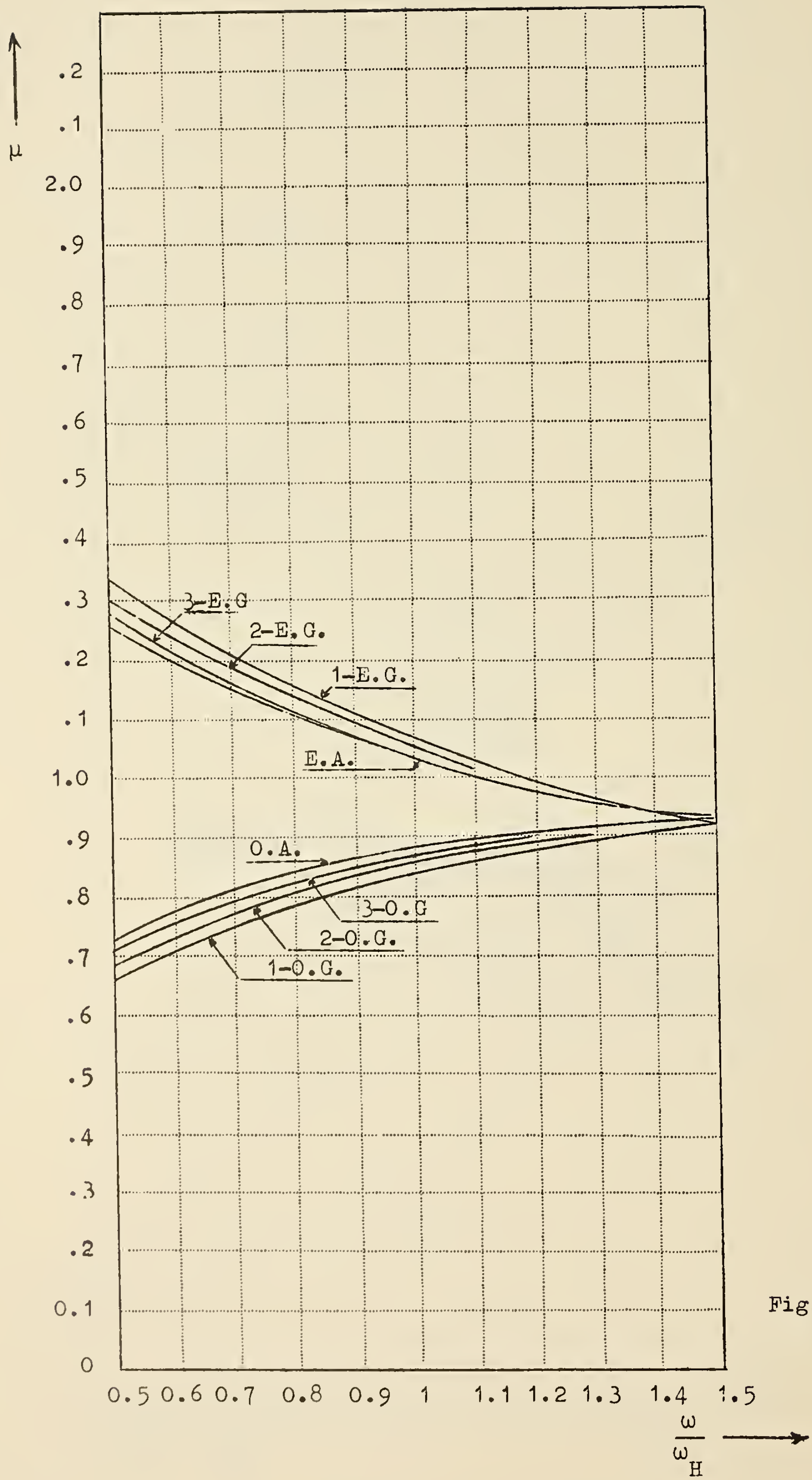

μ
.2
.1
2.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
1.0
.9
.8
.7
.6
.5
.4
.3
.2
0.1
0
3-E.G
2-E.G.
1-E.G.
E.A.
O.A.
3-O.G
2-O.G.
1-O.G.
0.5 0.6 0.7 0.8 0.9 1 1.1 1.2 1.3 1.4 1.5
$\frac{\omega}{\omega_H}$

Fig. N° 10

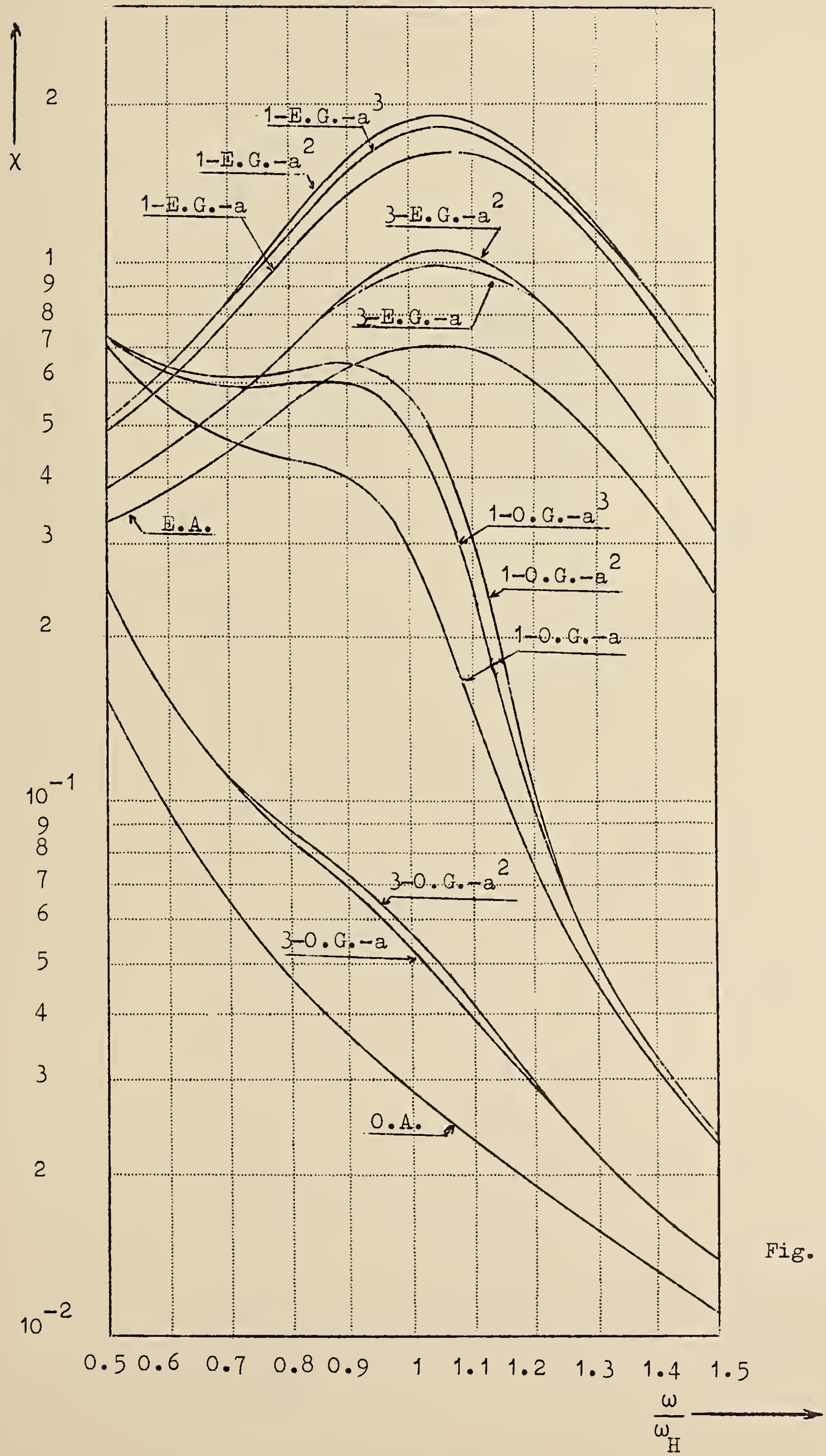
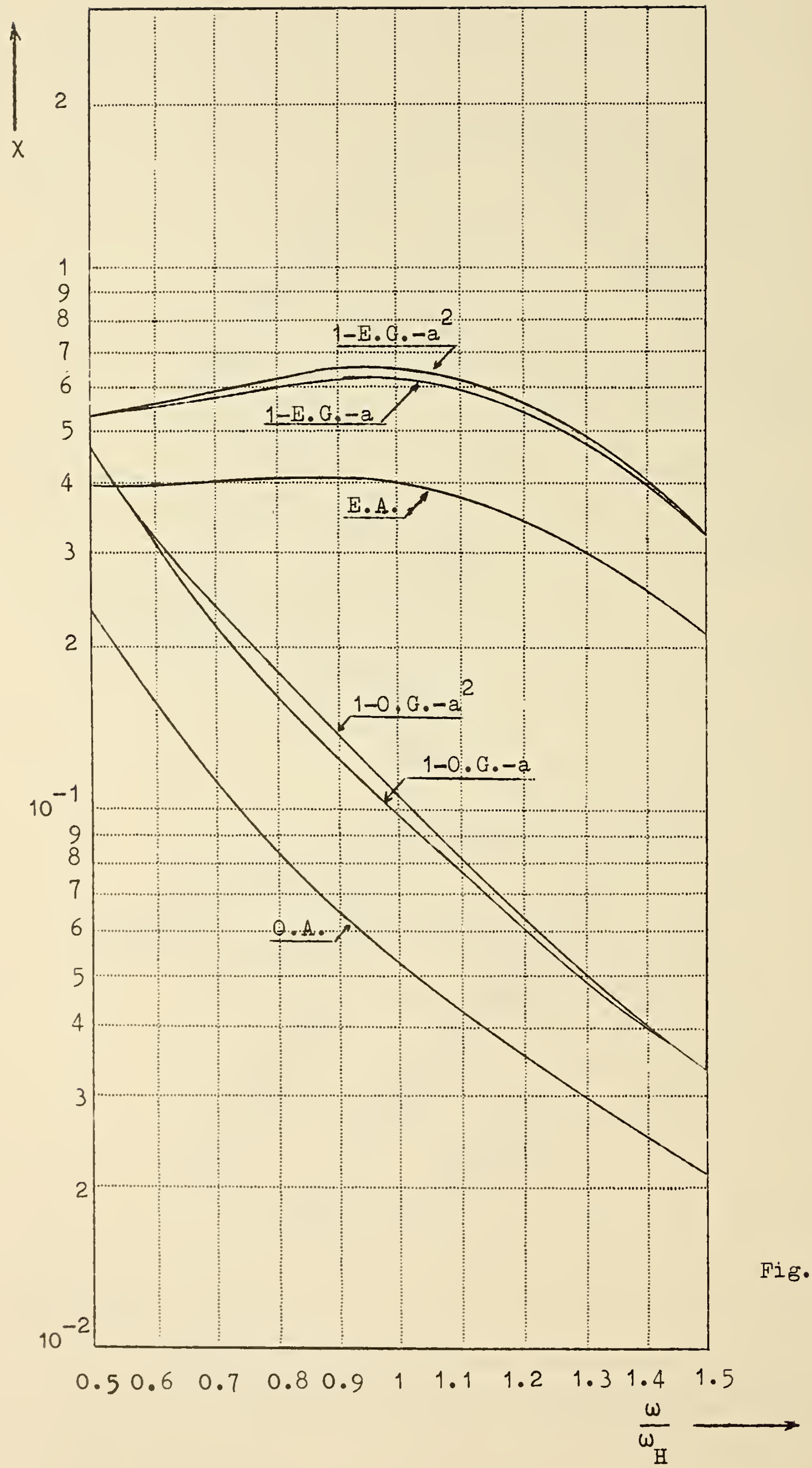

χ
2
1
9
8
7
6
5
4
3
2
10^{-1}
9
8
7
6
5
4
3
2
10^{-2}
0.5 0.6 0.7 0.8 0.9 1 1.1 1.2 1.3 1.4 1.5
$\frac{\omega}{\omega_H}$
1-E.G.-a^3
1-E.G.-a^2
1-E.G.-a
3-E.G.-a^2
3-E.G.-a
E.A.
1-O.G.-a^3
1-O.G.-a^2
1-O.G.-a
3-O.G.-a^2
3-O.G.-a
O.A.

Fig. Nº 11



X
2
1
9
8
7
6
5
4
3
2
10^{-1}
9
8
7
6
5
4
3
2
10^{-2}
1-E.G.-a^2
1-E.G.-a
E.A.
1-O.G.-a^2
1-O.G.-a
O.A.
0.5 0.6 0.7 0.8 0.9 1 1.1 1.2 1.3 1.4 1.5
$\frac{\omega}{\omega_H}$

Fig. N° 12

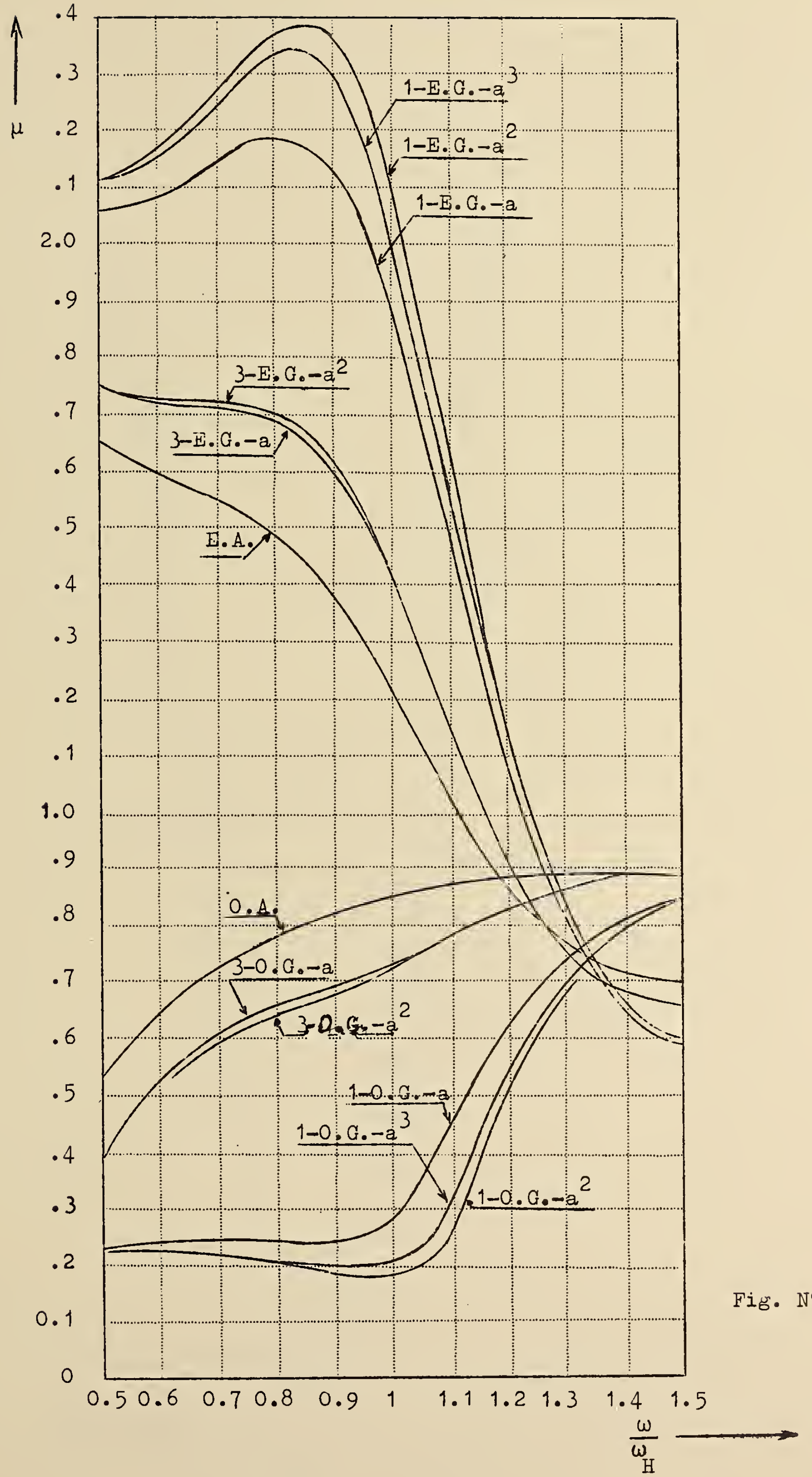

μ
.4
.3
.2
.1
2.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
1.0
.9
.8
.7
.6
.5
.4
.3
.2
0.1
0
1-E.G.-a3
1-E.G.-a2
1-E.G.-a
3-E.G.-a2
3-E.G.-a
E.A.
O.A.
3-O.G.-a
3-O.G.-a2
1-O.G.-a
1-O.G.-a3
1-O.G.-a2
0.5 0.6 0.7 0.8 0.9 1 1.1 1.2 1.3 1.4 1.5
ω/ωH

Fig. N° 13

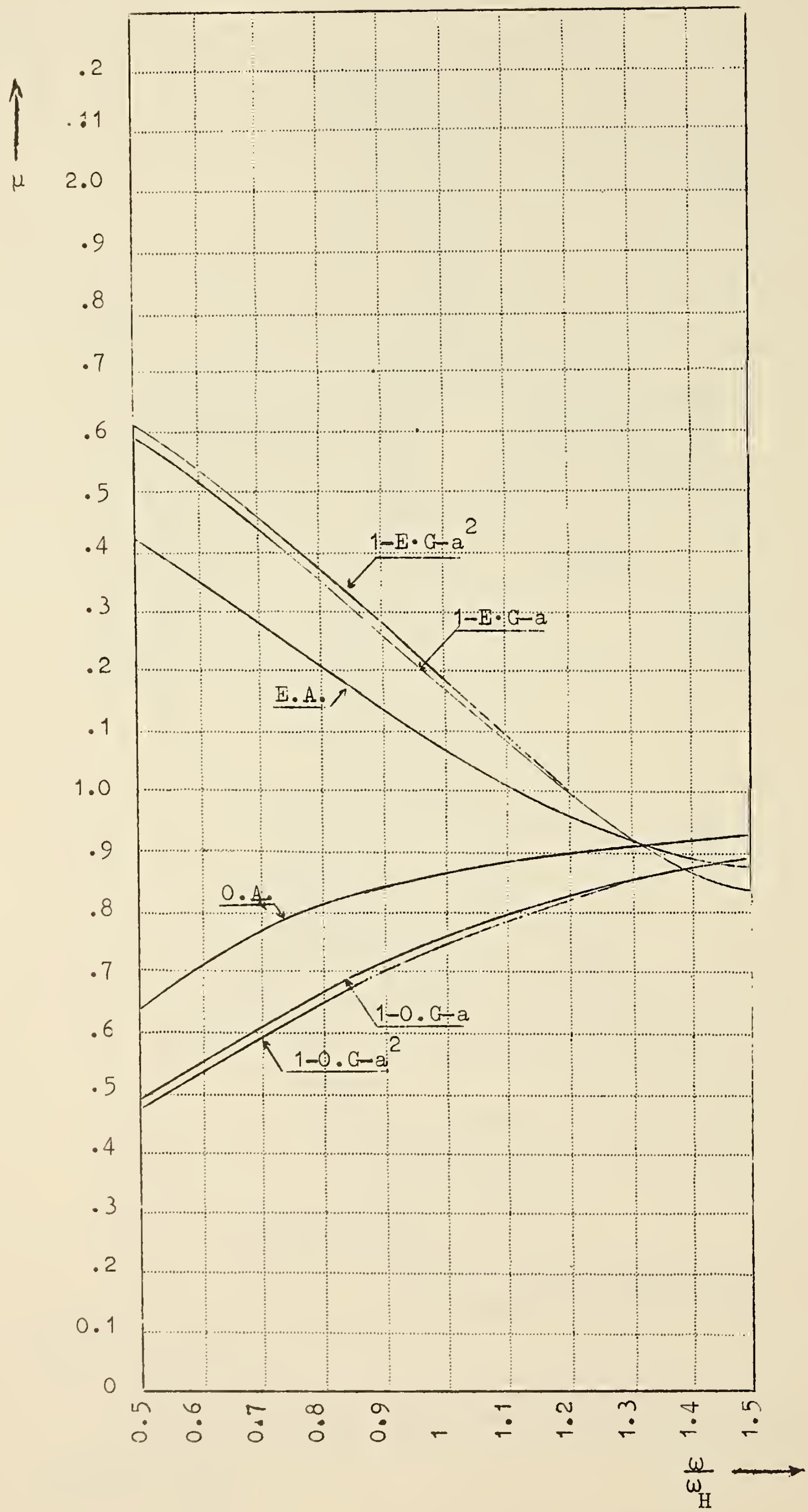

μ
.2
.11
2.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
1.0
.9
.8
.7
.6
.5
.4
.3
.2
0.1
0
1-E·G-a²
1-E·G-a
E.A.
O.A.
1-O.G-a
1-O.G-a²
0.5
0.6
0.7
0.8
0.9
1
1.1
1.2
1.3
1.4
1.5
$\frac{\omega}{\omega_H}$

Fig. N° 14

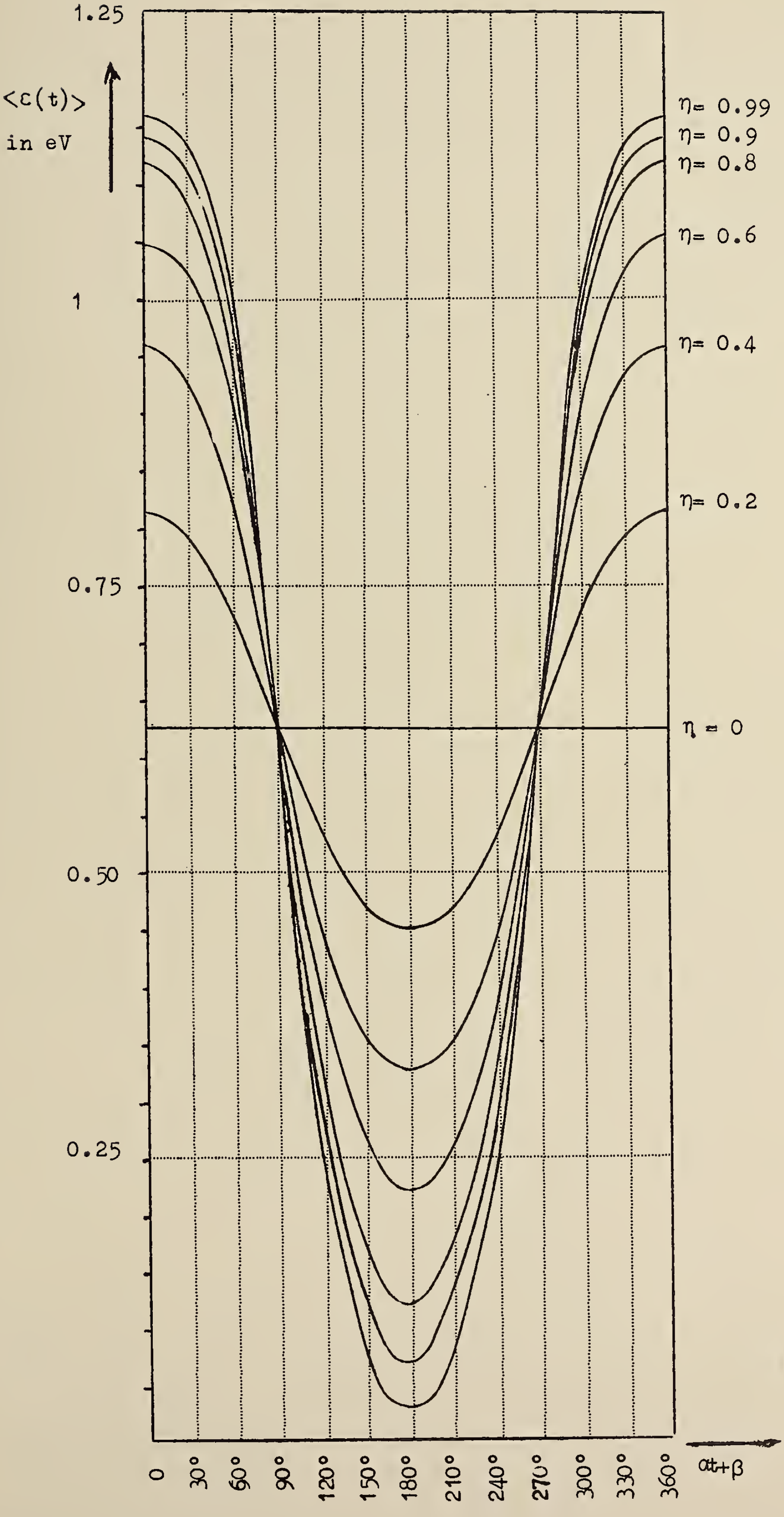

1.25
<ε(t)>
in eV
1
0.75
0.50
0.25
η= 0.99
η= 0.9
η= 0.8
η= 0.6
η= 0.4
η= 0.2
η = 0
0
30°
60°
90°
120°
150°
180°
210°
240°
270°
300°
330°
360°
αt+β

Fig. N° 15

DEPARTMENT OF COMMERCE ★ NATIONAL BUREAU OF STANDARDS ★